Afrika Statistika

Afrika Statistika
Vol. 13 (2), 2018, pages 1779 –1794.
DOI: http://dx.doi.org/10.16929/as/1779.133

ISSN 2316-090X

A Bayesian analysis of a change in the mean of independent normal sequence with contaminated observation

Abdeldjalil Slama [1,*] **and Hocine Fellag**[2]

[1] Department of Mathematics and Computer Science, University of Adrar, Adrar, Algeria.
[2] Department of Mathematics, Faculty of Sciences, University of Tizi Ouzou, Tizi-Ouzou, Algeria.

Received: July, 29 2018; Accepted: October 29, 2018

Abstract. In this paper, we consider a Bayesian analysis of a change in the mean of independent gaussian samples in the presence of a single outlier. An unconditional Bayesian significance test for testing change versus no change is performed under consideration of non informative prior distribution of the parameters. From a numerical study using the Gibbs sampler algorithm, the effect of a contaminated observation on the performance of the Bayesian significance test of change is studied.

Key words: Gaussian models; Change-point; HPD region sets; p-value; Outliers.
AMS 2010 Mathematics Subject Classification Objects : 91B84; 62F15; 62F03.

*Corresponding author Abdeldjalil Slama: slama_dj@yahoo.fr
Hocine Fellag : hfellag@yahoo.com

Résumé. Dans ce papier, nous considérons une analyse Bayésienne d'un changement dans la moyenne des échantillons gaussiens indépendants en présence d'une seule valeur aberrante. Tous les paramètres sont supposés inconnus. En considérant une distribution a priori non informative pour les paramètres, un test de signification bayésien inconditionnel pour tester le changement est proposé. En utilisant l'algorithme de l'échantillonneur de Gibbs, une étude numérique est réalisée pour examiner l'effet de la présence d'une observation contaminée sur la performance du test de signification bayésien.

1. Introduction and preliminaries

Suppose we have the observations $(x_1, \cdots, x_n)$ based on the following change point model :

$$\begin{cases} X_i = \phi_0 + \varepsilon_i \text{ if } \; i = 1, 2, \cdots, m \\ X_i = \phi_1 + \varepsilon_i \text{ if } i = m+1, \cdots, n \end{cases} \tag{1}$$

where the ε_i are normal random independent errors with mean zero and unknown constant variance σ^2. ϕ_0 and ϕ_1 are real unknown constants which represent the means of the variables X_i before and after the change-point m. n being the size of the sample.

The change point problems have been an active research area with a variety of applications. A change point, which is generally the effect of an external event on the phenomenon of interest, may be represented by a change in the structure of the model or simply by a change of the value of some parameters. Since Page (1954, 1955) who developed a cumulative sum (Cusum) test to detect a location change, considerable attention has been given to this problem with a variety of application and in a variety of settings, For a review, see Csorgo and Horvath (1997, 1987), Brodsky and Darkhovsky (1993) and Liu *et al.* (2008), Jandhyala *et al.* (2013).

In the Bayesian context, the problem of detecting a change was studied by many authors. We can cite Chernoff and Zacks (1964), Kander and Zacks (1966), whose aim was to detect the change in the mean of normal random variables. Kim (1991), proposed a Bayesian significance test for stationarity of a regression equation using the highest posterior density credible set. From a Monte Carlo simulation study, he showed that the Bayesian significance test has stronger power than the Cusum and the Cusum of squares tests suggested by Brown *et al.* (1975).

Slama (2014a) examined the effect of correlation on the performance of the Bayesian significance test derived under the assumption of no correlation. By numerical studies, he showed that the Bayesian significance test based on the HPD region is sensitive to the correlation in the data.Recently, Slama and Saggou (2017) considered a Bayesian analysis of a possible change in the parameters of autoregressive time series of known order p, AR(p). For a review of important results in parametric change point(s) analysis, we can see the monograph of Chen *et al.* (2011) and the references contained therein.

In many applications, however, the observations can be contaminated by the presence of outliers; i.e. observations that deviate significantly from the majority of observations. This may due to errors in the collection, measurement and processing of the data, or to some unexpected event influencing the phenomenon under study. Outliers may have a significant impact on the results of standard methodology for time series analysis, therefore it is important to detect them, estimate their effects and undertake the appropriate corrective actions. For example, the impact of outliers on parameter estimation has been studied by Pena (1990), on autoregressive moving-average (ARMA) identification by Deutsch *et al.* (1990), and the effects on forecasts are addressed by Ledolter (1989) and Chen and Liu (1993). For more details see Battaglia *et al.* (2005) and the references contained therein.

Also, the Bayesian analysis of models in the presence of outliers has been considered by several authors. Verdinelli and Wasserman (1991) considered the Bayesian analysis of outlier models, they showed that the Gibbs sampler brings considerable conceptual and computational simplicity to the problem of calculating posterior marginals. Nasiri (2010) proposed the maximum likelihood estimator and the Bayes estimators of the parameters of the generalized exponential distribution in the presence of outliers. The performances of these estimators have been compared to each other, and they concluded that the Bayes estimators are better than the maximum likelihood estimator. Belkacem and Fellag (2012) studied the impact of an outlier on the performance of the Bayesian estimation of the change point in independent gaussian samples. Slama (2014b) studied the impact of an outlier on the performance of the Bayesian significance test based on the HPD regions sets. The position and the magnitude of the contamination are assumed to be known. The simulation study showed that the Bayesian significance test based on the HPD region is insensitive to the presence of a single outlier. Recently, Gupta and Singh (2017) considered the classical and Bayesian estimation of the parameters of Weibull distribution in presence of outlier. A simulation study has been conducted to compare the performance of the classical and Bayesian methods of estimation.

In this paper, we extend the work in Slama (2014b) to the cases where the position and the magnitude of the contamination are unknown, and our aim is to study the effects of the presence of an additive outlier on the performance of the Bayesian significance test of change in the mean of independent gaussian samples. The posterior estimates of the change point, the position and the magnitude of the contamination, and the others parameters of the model are given. By extensive simulations, the posterior estimates of parameters are determined. As well, we compare the rejection rates of the null using the unconditional test with those determined by the test taken under the assumption of non contamination.

The rest of the paper is organized as follows. Section 2 presents the models assumed in the present paper. In section 3 we present the Bayesian analysis of the parameters of the model and the Bayesian significance test of change. A simulations results are given in section 4. Section 5 is our conclusion.

Assume that there exists a position k, $k \in \{1, 2, \cdots, n\}$, such that $(y_1, \cdots, y_n)$ are possible observations from the model,

$$\begin{cases} Y_k = X_k + \xi \\ Y_i = \quad X_i \quad \forall i \in \{1, 2, \cdots, n\} \quad \text{with} \quad i \neq k, \end{cases} \tag{2}$$

where the constant ξ is the magnitude of the contamination which occurs at time k.

We suppose that the contaminations occurs before the change point m, i.e., $k \in \{1, \cdots, m\}$. The model (2) is written as follows :

$$\begin{cases} Y_i = X_i = \phi_0 + \varepsilon_i & i = 1, \cdots, k-1 \quad \text{and} \quad i = k+1, \cdots, m \\ Y_k = X_k + \xi = \phi_0 + \xi + \varepsilon_k & \\ Y_i = X_i = \phi_1 + \varepsilon_i & i = m+1, \cdots, \quad n, \end{cases} \tag{3}$$

where $m \in \{1, \cdots, n-1\}$, $k \in \{1, \cdots, m\}$, $\phi_0, \phi_1 \in \mathbb{R}$, ($\phi_0 \neq \phi_1$), $\xi \in \mathbb{R}$, and $\varepsilon_i \sim N(0, \sigma^2)$, ($\sigma > 0$), with m, k, ξ, ϕ_0, ϕ_1 and σ are unknown parameters.

The likelihood function based on the observations $y = (y_1, y_2, \cdots, y_n)$ is then,

$$l(y \mid \theta) \propto \quad \sigma^{-n} \exp\left\{ -\frac{1}{\sigma^2} \left[\sum_{\substack{i=1 \\ i \neq k}}^{m} (y_i - \phi_0)^2 + (y_k - (\phi_0 + \xi))^2 + \sum_{i=m+1}^{n} (y_i - \phi_1)^2 \right] \right\}. \tag{4}$$

2. Bayesian analysis

One has a parameter set $\theta = (m, k, \xi, \phi_0, \phi_1, r)$ where $r = 1/\sigma^2$. Since prior knowledge of $\theta' = (\phi_0, \phi_1, \xi, r)$ is often vague or diffuse, we employ a diffuse prior for θ'. The parameters (k, m), ϕ_0, ϕ_1, ξ and r are assumed independent. we suppose that $\pi(k, m) \propto \dfrac{1}{m}$, $k \in \{1, \cdots, m\}$, $m \in \{1, \cdots, n-1\}$. The prior distribution of θ is, therefore

$$\pi(\theta) \propto \pi(k, m).\frac{1}{r} = \frac{1}{m}.\frac{1}{r}, \tag{5}$$

Note that the functional forms $\pi(.)$ and $\pi(. \mid .)$ represent a prior and a posterior distribution, respectively.
The posterior distribution of θ, obtained by combination of (4) and (5) is

$$\pi(\theta \mid y) \propto \quad m^{-1} r^{\frac{n}{2}-1} \exp\left\{-\frac{r}{2}\left[\sum_{\substack{i=1\\ i\neq k}}^{m}(y_i-\phi_0)^2+(y_k-(\phi_0+\xi))^2+\sum_{i=m+1}^{n}(y_i-\phi_1)^2\right]\right\}. \quad (6)$$

We build an inference about testing the hypothesis, that is, to test whether or not a change point occurs in the mean of in independent gaussian samples in the presence of a single outlier, and we will study the effect of the presence of an outlier on the performance of the test.

The null hypothesis H_0, that there is no change in the parameters of model (1), is

$$H_0 : \delta = \phi_1 - \phi_0 = 0 \quad \text{against} \quad H_1 : \delta = \phi_1 - \phi_0 \neq 0$$

The proposed test is based on the posterior distribution of the shift $\delta = \phi_1 - \phi_0$. The hypothesis meaning "non change" is equivalent to $H_0' : m = n$ and H_1 is equivalent to $H_1' : m \neq n$.

The following theorem gives the posterior distribution of the magnitude of the shift in the mean δ, the magnitude of the contamination ξ, the position of the contamination k and the change point m.

Theorem 1.
(1) Given m, k, ξ and ϕ_0, the conditional posterior distribution of δ is:

$$\pi(\delta \mid m,k,\xi,\phi_0,y) \propto \left\{1+\frac{(n-m)\left(\delta-\widehat{\delta}(m,\phi_0)\right)^2}{(n-1)S_1^2(m,k,\xi,\phi_0)}\right\}^{-\frac{n}{2}}, \quad (7)$$

where

$$\widehat{\delta}(m,\phi_0) = \frac{\sum_{i=m+1}^{n}(y_i-\phi_0)}{n-m},$$

$$S_1^2(m,k,\xi,\phi_0) = \frac{SS(m,k,\xi,\phi_0)}{(n-1)}$$

and

$$SS(m,k,\xi,\phi_0) = \sum_{\substack{i=1\\ i\neq k}}^{n}(y_i-\phi_0)^2+(y_k-(\phi_0+\xi))^2-\frac{\left[\sum_{i=m+1}^{n}(y_i-\phi_0)\right]^2}{n-m}, \quad (8)$$

which is the Student t distribution with location parameter $\widehat{\delta}(m,\phi_0)$, precision $\dfrac{n-m}{S_1^2(m,k,\xi,\phi_0)}$, and $(n-1)$ degrees of freedom. Equivalently, the quantity

$$t(\delta) = \frac{(n-m)^{\frac{1}{2}}\left(\delta-\widehat{\delta}(m,\phi_0)\right)}{S_1(m,k,\xi,\phi_0)} \quad (9)$$

is distributed a posteriori as a conditional standard Student t distribution with $(n-1)$ degrees of freedom given m and ϕ_0.

(2) Given m, k, ϕ_0 and δ, the conditional posterior distribution of ξ is:

$$\pi(\xi \mid m, k, \phi_0, \delta, y) \propto \left\{ 1 + \frac{\left(\xi - \widehat{\xi}(\phi_0)\right)^2}{(n-1)S_3^2(m,k,\phi_0,\delta)} \right\}^{-\frac{n}{2}}, \tag{10}$$

where,

$$\widehat{\xi}(\phi_0) = y_k - \phi_0, \qquad S_3^2(m,k,\phi_0,\delta) = \frac{SS_3(m,k,\phi_0,\delta)}{n-1}$$

and,

$$SS_3(m,k,\phi_0,\delta) = \sum_{\substack{i=1 \\ i\neq k}}^{m} (y_i - \phi_0)^2 + \sum_{m+1}^{n} (y_i - \delta - \phi_0)^2, \tag{11}$$

which is the Student t distribution with location parameter $\widehat{\xi}(\phi_0)$, precision $\frac{1}{S_3^2(m,k,\phi_0,\delta)}$, and $(n-1)$ degrees of freedom.

(3) Given m, ξ and ϕ_0, the conditional posterior distribution of k is:

$$\pi(k \mid m, \xi, \phi_0, y) \propto SS(m,k,\xi,\phi_0)^{-\frac{n-1}{2}} \tag{12}$$

where $SS(m,k,\xi,\phi_0)$ is given in (8).

(4) Given k, ξ, and ϕ_0, the conditional posterior distribution of m is:

$$\pi(m \mid k, \xi, \phi_0, y) \propto m^{-1}(n-m)^{-\frac{1}{2}} SS(m,k,\xi,\phi_0)^{-\frac{n-1}{2}},$$

where $SS(m,k,\xi,\phi_0)$ is given in 8.

Proof of Theorem 1. By transforming the parameter set $\Theta = (m,k,\xi,\phi_0,\phi_1,r)$ into $\Phi = (m,k,\xi,\phi_0,\delta)$, we can form the posterior distribution of Φ; that is,

$$\begin{aligned} \pi(\Phi \mid y) &= \int_r \pi(m,k,\xi,\phi_0,\delta+\phi_0,r/y)dr \\ &\propto m^{-1}\left[\sum_{\substack{i=1\\ i\neq k}}^{m}(y_i-\phi_0)^2 + (y_k-(\phi_0+\xi))^2 + \sum_{m+1}^{n}(y_i-\delta-\phi_0)^2\right]^{-\frac{n}{2}} \\ &\propto m^{-1}\left[\sum_{\substack{i=1\\ i\neq k}}^{n}(y_i-\phi_0)^2 + (y_k-(\phi_0+\xi))^2 - \frac{\left[\sum_{m+1}^{n}(y_i-\phi_0)\right]^2}{n-m}\right. \\ &+ \left.(n-m)\left(\delta - \widehat{\delta}(m,\phi_0)\right)^2\right]^{-\frac{n}{2}}. \end{aligned} \tag{13}$$

We conclude the proof of the following points :

(i) By application of Bayes theorem, the posterior conditional distribution of δ is obtained as given in (7).

(ii) By integration with respect of δ, we obtained the joint posterior distribution of m, k, ξ and ϕ_0:

$$\pi(m,k,\xi,\phi_0 \mid y) \propto m^{-1}(n-m)^{-\frac{1}{2}} SS(m,k,\xi,\phi_0)^{-\frac{n-1}{2}}. \tag{14}$$

where $SS(m,k,\xi,\phi_0)$ is given in (8).

(iii) The conditional posterior distribution of ξ and k are obtained by the successive application of Bayes' theorem on posterior distribution given in (13).

(iv) By application of Bayes theorem, the posterior conditional distribution of m is :

$$\pi(m \mid k,\xi,\phi_0,y) \propto m^{-1}(n-m)^{-\frac{1}{2}} SS(m,k,\xi,\phi_0)^{-\frac{n-1}{2}}$$

where $SS(m,k,\xi,\phi_0)$ is given in (8). □

The following Lemma gives the posterior distribution of the mean before the change point ϕ_0.

Lemma 1.
(1) Given m, k, ξ and δ, the conditional posterior distribution of ϕ_0 is:

$$\pi\left(\phi_0 \mid m,k,\xi,\delta,y\right) \propto \left\{1 + \frac{n\left(\phi_0 - \widehat{\phi_0}(m,k,\xi,\delta)\right)^2}{(n-1)S_2^2(m,k,\xi,\delta)}\right\}^{-\frac{n}{2}}, \tag{15}$$

where,

$$\widehat{\phi_0}(m,k,\xi,\delta) = \frac{b(m,k,\xi,\delta)}{n},$$

and

$$S_2^2(m,k,\xi,\delta) = \frac{1}{(n-1)}\left[a(m,k,\xi,\delta) - \frac{b^2(m,k,\xi,\delta)}{n}\right].$$

with,

$$a(m,k,\xi,\delta) = \sum_{\substack{i=1\\ i\neq k}}^{m} y_i^2 + (y_k-\xi)^2 + \sum_{m+1}^{n}(y_i-\delta)^2,$$

and

$$b(m,k,\xi,\delta) = \sum_{\substack{i=1\\ i\neq k}}^{m} y_i + (y_k-\xi) + \sum_{m+1}^{n}(y_i-\delta).$$

Thus is the Student t distribution with location parameter $\widehat{\phi_0}(m,k,\xi,\delta)$, with precision $\dfrac{n}{S_2^2(m,k,\xi,\delta)}$, and $(n-1)$ degrees of freedom.

Proof. The conditional posterior distributions in (15) is obtained by application of Bayes' theorem on posterior distribution given in (13). □

The unconditional posterior distributions of $t(\delta)$ is,

$$\pi(t(\delta) \mid y) = \sum_m \sum_k \int_\xi \int_{\phi_0} \pi(t(\delta) \mid m, k, \xi, \phi_0, y)\pi(\phi_0 \mid m, k, \xi, y) \\ \times \pi(\xi \mid m, k, y)\pi(k \mid m, y)\pi(m \mid y) \tag{16}$$

One defines the highest posterior density credible sets of $t(\delta)$. The credible set will be used to define the unconditional p-value and therefore an unconditional test.

Given m, k, ξ and ϕ_0, the $(1-\alpha)$-credible set for $t(\delta)$ is defined as:

$$C_\delta = \left\{t(\delta)/|t(\delta)| < t_{\alpha|2}(n-1)\right\},$$

where $t_{\alpha|2}(n-1))$ is the $(1-\alpha/2)$th quantile of a t-distribution with $(n-1)$ degrees of freedom. Hence, given m, k, ξ and ϕ_0, the decision rule for H_0, is to reject if $t(0) \in \overline{C_\delta}$, where $\overline{C_\delta}$ is the complement of C_δ.

The unconditional p-value of H_0, therefore, is calculated from (16) to yield:

$$\begin{aligned} P_{\delta=0|y} =& 2\sum_m \sum_k \left[\int_\xi \left(\int_{\phi_0} \left[1 - \mathcal{T}_{n-1}(| t(0) |)\right] \right.\right. \\ & \times \pi(\phi_0 \mid m, k, \xi, y)d\phi_0) \,\pi(\xi \mid m, k, y)d\xi]\, \pi(k \mid m, y)\pi(m \mid y), \\ & = 2E_m E_k E_\xi E_{\phi_0}\left[1 - \mathcal{T}_{n-1}(| t(0) |)\right]. \end{aligned} \tag{17}$$

where $\mathcal{T}_{n-1}$ is the cumulative density function of the standard Student t distribution with $(n-1)$ degrees of freedom, and the expectations E_m, E_k, E_ξ and E_{ϕ_0} are taken with respect to m, k, ξ and ϕ_0.
Our test, therefore, rejects H_0 unconditionally, if $P_{\delta=0|y}$ falls below α, $(0 < \alpha < 1)$. Which define the unconditional Bayesian significance test (UncBST) of H_0.

Given ξ, the conditional p-value of H_0, is:

$$\begin{aligned} P_{\delta=0|\xi,y} =& 2\sum_m \sum_k \left[\left(\int_{\phi_0} \left[1 - \mathcal{T}_{n-1}(| t(0) |)\right]\pi(\phi_0 \mid m, k, \xi, y)d\phi_0\right)\right] \\ & \times \pi(k \mid m, y)\pi(m \mid y), \\ & = 2E_m E_k E_{\phi_0}\left[1 - \mathcal{T}_{n-1}(| t(0) |)\right]. \end{aligned} \tag{18}$$

and so, H_0 will be rejected conditionally to ξ, if $P_{\delta=0|\xi,y} < \alpha$. Thus, define the Bayesian significance test of H_0 conditionally on ξ (ConBST).

The quantities given in (17) and (18) will be evaluated numerically by the Gibbs sampler algorithm by using the conditional posterior distributions given in in Theorem 1) and Lemma 1). The Gibbs sampler is a Markovian updating scheme enabling one to obtain samples from a joint distribution via iterated sampling from full conditional distributions. Detailed investigation of the Gibbs sampler applied to general Bayesian calculation is given by Gelfand and Hills (1990) and Gelfand and Smith (1990).

Remark 1. The contamination can occur after the change-point m. In this case, the same methodology than above can be adapted to determine the unconditional p-value of H_0.

The model is written as follows:

$$\begin{cases} Y_i = X_i = \phi_0 + \varepsilon_i & i = 1, \cdots, m \\ Y_k = X_k + \xi = \phi_1 + \xi + \varepsilon_k & \\ Y_i = X_i = \phi_1 + \varepsilon_i & i = m+1, \cdots, k-1 \\ & i = k+1, \cdots, n \,. \end{cases} \tag{19}$$

where m, k, ξ, ϕ_0, ϕ_1 and σ are unknown parameters.

The prior distribution of $\theta = (m, k, \xi, \phi_0, \phi_1, r)$ is,

$$\pi(\theta) \propto \pi(k, m).\frac{1}{r} = \frac{1}{n-m}.\frac{1}{r}, \tag{20}$$

Then, the posterior distribution for the parameter θ is,

$$\pi(\theta \mid y) \propto \quad (n-m)^{-1} r^{-\frac{n}{2}-1} \exp\left\{ -\frac{r}{2} \left[\sum_{i=1}^{m} (y_i - \phi_0)^2 + (y_k - (\phi_1 + \xi))^2 + \sum_{\substack{i=m+1 \\ i \neq k}}^{n} (y_i - \phi_1)^2 \right] \right\} \tag{21}$$

3. Simulation study:

Simulation has been used to study the effect of contaminated observations by a single outlier on the Bayesian significance test based on the highest posterior density credible set.

We simulated a sample from the model (2) with $n = 200$, $m = 120$, $\sigma = 1$, $k = 80$ and for different values of ϕ_0, ϕ_1 and ξ. The Simulated observations are represented in the following figure (1) and (2).
From these observations, by the application of the Gibbs sampler algorithm with 2000 replicates, we approximate the unconditional p-values for the hypothesis $H_0 : \delta = 0$; $P_{\delta=0|y}$, the conditional p-values on ξ for $H_0 : \delta = 0$; $P_{\delta=0|\xi,y}$, the posterier density function of m, the posterior distribution of δ and of ξ, and the conditional posterior distribtion of k given m. The results are given in tables (1) - (6) and in figures (3) - (4) .

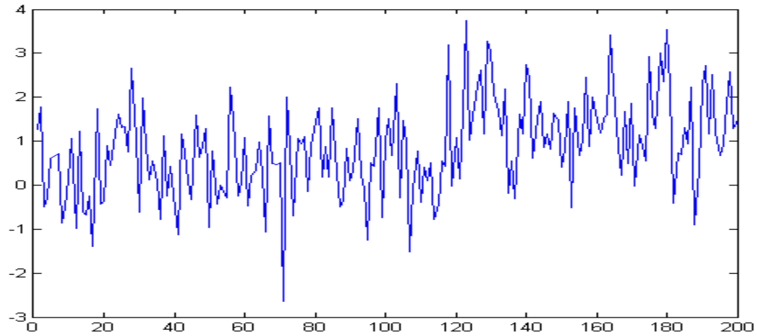

Fig. 1. A Simulated observations with $\phi_0 = 0.5$, $\phi_1 = 1.5$, $k = 80$ and $\xi = 0$

The figures (3) - (4) illustrate the posterior density function of m with $\xi = 3$ and $\xi = 0$ respectively. We can readily see from these figures that the posterior mode is the true value of m. This goes in the same direction as the result of Belkacem and Fellag (2012), who have found that, if the sample size is high, the presence of an outlier has not a significant impact on the Bayesian procedure of estimation of the change-point in independent gaussian samples.

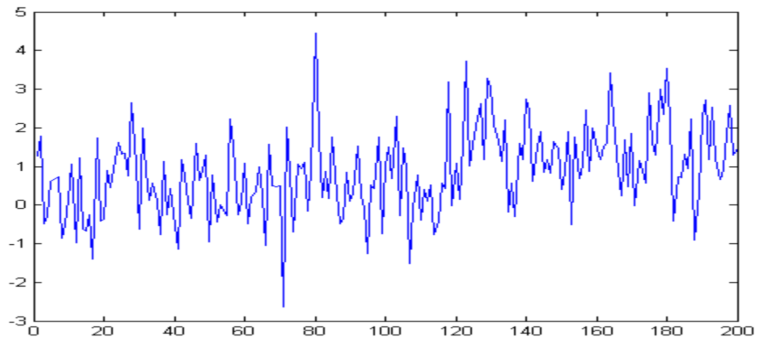

Fig. 2. A Simulated observations with $\phi_0 = 0.5$, $\phi_1 = 1.5$, $k = 80$ and $\xi = 3$

The values of $P_{\delta=0|y}$ and of $P_{\delta=0|\xi=0,y}$ in the table (1) show that, for $\delta = 0.5$, the test rejects unconditionally and conditionally on $\xi = 0$ the hypothesis H_0 at significance level $\alpha = 0,01$, for all values of ξ, and for $\delta = 0,25$, the test rejects unconditionally and conditionally H_0 at significance level $\alpha = 0,10$, for all values of ξ. For $\delta = 0$, the hypothesis H_0 can not be rejected at 10%, for all values of ξ. Recall that the mean is not changed. Likewise, It is observed that the p-values taken under the assumption of non contamination; $P_{\delta=0|\xi=0,y}$, are significantly the same that the unconditional p-values; $P_{\delta=0|y}$, for all values of ξ. It means that, the Bayesian significance test based on the HPD regions for detecting a change in the mean of independent Gaussian samples is insensitive to the presence of single outlier (see table (1)).

		$\delta = 1 - 0.5 = 0.5$	$\delta = 1 - 0.75 = 25$	$\delta = 1 - 1 = 0$
$\xi = 0$	$P_{\delta=0\|y}$	0.0016	0.0710	0.1460
	$P_{\delta=0\|\xi=0,y}$	0.0011	0.0755	0.1472
$\xi = 2$	$P_{\delta=0\|y}$	0.0052	0.0733	0.1322
	$P_{\delta=0\|\xi=0,y}$	0.0054	0.0655	0.1524
$\xi = 4$	$P_{\delta=0\|y}$	0.0008	0.0831	0.1414
	$P_{\delta=0\|\xi=0,y}$	0.0009	0.0948	0.1426
$\xi = 8$	$P_{\delta=0\|y}$	0.0215	0.0813	0.1377
	$P_{\delta=0\|\xi=0,y}$	0.0220	0.0819	0.1347

Table 1. The unconditional p-values for H_0 and conditional p-values on $\xi = 0$ for different values of ξ and δ.

Using the same generated series, various simulations were run for various values of ξ and δ. Tables (2) - (4) illustrates the posterior estimates of parameters, m, k, ξ and δ.

The posterior mean and median of k and δ estimate nearly well the true value of k and δ, while the posterior mean of m and ξ presents some bias, (see Tables (2) - (3)). But, for a large shift in the mean $(\delta = 3)$, the posterior mean of the change-point m is equal to the true value of m (see table (3)). Note also that, all the 95% highest posterior density (HPD) intervals of the parameters contain the true value considered. Table (4) shows that, the 95% highest posterior density (HPD) interval of the parameter k; the position of contamination; contains nearly all possible values of k. Notice that the magnitude of the contamination ξ is equal to zero.

Parameters	True values	Median	Mean(SD)	2.5%	97,5%
m	120	120	119.712(1.6829)	116	122
k	80	80^*	$80.088^*(7.6457^*)$	71^*	103^*
ξ	3	3.9171	3.8086(1.3765)	0.8079	6.1334
δ	2	2.0261	2.0306(0.1447)	1.7376	2.3237

Table 2. Posterior estimates of parameters m, k, ξ and δ when $m = 120$, $k = 80$, $\xi = 3$ and $\delta = 2.5 - 0.5 = 2$. (The values with star indication are calculated conditionally to m=120).

Parameters	True values	Median	Mean(SD)	2.5%	97,5%
m	120	120	120.259(0.6192)	119	121
k	80	80^*	$79.94^*(6.5820^*)$	71.75^*	80^*
ξ	3	3.9558	3.8110(1.4219)	-0.0656	5.9345
δ	3	3.0204	3.0241(0.1430)	2.7493	3.3087

Table 3. Posterior estimates of parameters m, k, ξ and δ when $m = 120$, $k = 80$, $\xi = 3$ and $\delta = 3.5 - 0.5 = 3$. (The values with star indication are calculated conditionally to m=120)

Parameters	True values	Median	Mean(SD)	2.5%	97,5%
m	120	120	119.736(1.6263)	116.75	122
k	80	71^*	$73.256^*(30.1024^*)$	9.75^*	118^*
ξ	0	-1.3581	-0.9293(2.4308)	-4.7639	3.7308
δ	2	2.0018	2.0049(0.1402)	1.7368	2.3018

Table 4. Posterior estimates of parameters m, k, ξ and δ when $m = 120$, $k = 80$, $\xi = 0$ and $\delta = 2.5 - 0.5 = 2$. (The values with star indication are calculated conditionally to m=120)

To illustrate the effect of contamination of the observation on the Bayesian significance test of change, we simulated 100 samples from the contaminated model (2) with $n = 200$, $m = 120$, $\delta = 0.5$, $\sigma = 1$, $k = 80$ and for different values of ξ, and we computed the rejection rates of H_0 by the Bayesian significance test conditionally on $\xi = 0$ (ConBST) and the unconditional Bayesian significance test of change (UncBST) at $\alpha\%$ significance levels. We consider three values of α, 0.01, 0.05 and 0.10. The results are given in the Tables (5), (6) and (7).

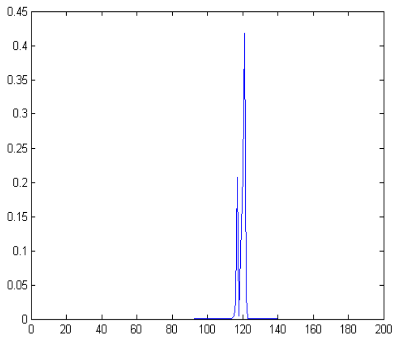

Fig. 3. Posterior density functions of m with $m = 120$, $k = 80$, $\delta = 2$ and $\xi = 3$

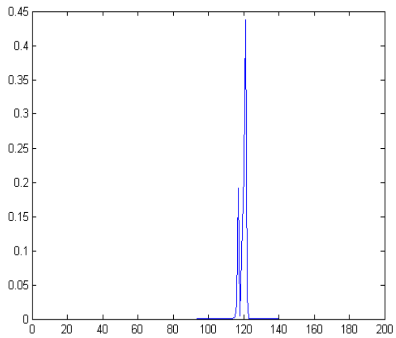

Fig. 4. Posterior density functions of m with $m = 120$, $k = 80$, $\delta = 2$ and $\xi = 0$

	ξ	0	2	4	8	10
UncBST	$\alpha = 0.01$	0,69	0,64	0,52	0,14	0,06
	$\alpha = 0.05$	0,97	0,88	0,87	0,50	0,30
	$\alpha = 0.10$	1,00	0,98	1,00	0,87	0,76
	$\alpha > 0.10$	0,00	0,02	0,00	0,13	0,24
ConBST	$\alpha = 0.01$	0,72	0,61	0,41	0,07	0,02
	$\alpha = 0.05$	0,94	0,88	0,80	0,32	0,20
	$\alpha = 0.10$	1,00	1,00	0,96	0,74	0,64
	$\alpha > 0.10$	0,00	0,00	0,04	0,26	0,36

Table 5. The rejection rates of H_0 by the conditionaly on $\xi = 0$ and inconditionaly Bayesian significance test at $\alpha\%$ level with $n = 200$. ConBST: Bayesian significance test conditionaly on $\xi = 0$, UncBST: Unconditional Bayesian significance test.

Tables (5)-(7) show that, for all values of n ($n = 200$, $n = 250$ and $n = 300$), the rejection rates for H_0, at $\alpha\%$ significance levels, are substantially the same for the two testing procedures considered (Unconditional Bayesian significance test(UncBST) and conditional Bayesian significance test (ConBST) given $\xi = 0$) for all values of the contamination ξ. For $n = 200$, the rejection rates at 10% level are more than 64% for the tow testing procedure and for all values of ξ. For $n = 250$, the rejection rates at 10% level are more than 86% for the tow testing procedure and for all values of ξ. Finaly, For $n = 300$, the rejection rates at 10% level are more than 97% for the tow testing procedure and for all values of ξ.

Thereby, take a large enough sample size has a positive effect on the Bayesian significance test of a change in the mean of a sequence of an independent normal random variables with contaminated observation, and makes the test insensitive

	ξ	0	2	4	8	10
UncBST	$\alpha = 0.01$	0,78	0,79	0,69	0,04	0,03
	$\alpha = 0.05$	0.99	0.94	0.96	0,34	0,20
	$\alpha = 0.10$	1,00	1,00	1,00	0,94	0,88
	$\alpha > 0.10$	0,00	0,00	0,00	0,06	0,12
ConBST	$\alpha = 0.01$	0,81	0,76	0,63	0,00	0,00
	$\alpha = 0.05$	0,93	0,94	0,90	0,25	0,14
	$\alpha = 0.10$	0,99	1,00	0,99	0,92	0,86
	$\alpha > 0.10$	0,01	0,00	0,01	0,08	0,14

Table 6. The rejection rates of H_0 by the conditionaly on $\xi = 0$ and inconditionaly Bayesian significance test at α% level with $n = 250$. ConBST: Bayesian significance test conditionaly on $\xi = 0$, UncBST: Unconditional Bayesian significance test.

	ξ	0	2	4	8	10
UncBST	$\alpha = 0.01$	0,18	0,11	0,07	0,00	0,00
	$\alpha = 0.05$	0,99	1,00	0,93	0,55	0,29
	$\alpha = 0.10$	1,00	1,00	1,001	1,00	0,98
	$\alpha > 0.10$	0,00	0,00	0,00	0,00	0,02
ConBST	$\alpha = 0.01$	0,18	0,19	0,05	0,00	0,00
	$\alpha = 0.05$	0,98	0,93	0,95	0,49	0,29
	$\alpha = 0.10$	1,00	1,00	1,00	0,99	0,97
	$\alpha > 0.10$	0,00	0,00	0,00	0,01	0,03

Table 7. The rejection rates of H_0 by the conditionaly on $\xi = 0$ and inconditionaly Bayesian significance test at α% level with $n = 300$. ConBST: Bayesian significance test conditionaly on $\xi = 0$, UncBST: Unconditional Bayesian significance test.

to the presence of a single outlier. However, the test becomes sensitive when the sample size becomes small.

4. Concluding remarks

This paper presented a Bayesian analysis of a change in the mean of independent gaussian samples in the presence of a single outlier under consideration of non-informative prior distribution of the parameters. All parameters are considered unknown. The use of non-informative prior distributions is motivated by the fact that, in general, the prior information is vague or unavailable. We have obtained the posterior estimates of the parameters and presented a Bayesian significance test of change in the mean. By numerical studies using the Gibbs sampler algorithm to obtain the required Bayesian estimations, we have shown that, with sufficiently large enough sample sizes, the Bayesian significance test based on the calculation of the p-values for a change in the mean of independent gaussian samples is insensitive to the presence of a single outlier. On the other hand, a study of the procedure robustness can be conducted by considering other prior distributions. For example, one may consider a normal distribution, conjugate prior distributions, or even a prior distribution where the parameters are not independent. Future research may be concerned with extension to other outlier types and to a wider class of gaussian models with change in the variance.

Acknowledgements. The authors would like to thank the Editor-in-Chief and the anonymous referees for their careful reading and suggestions that led to a substantial improvement of the presentation of the paper.

References

Battaglia, F., Orfei, L. (2005) Outlier detection and estimation in nonlinear time series. *Journal of Time Series Analysis* 26, 107-121.

Belkacem, C. and Fellag, H. (2012) Bayesian change-point estimation in the presence of a single outlier, *Journal Afrika Statistika*. Vol. 7, pages 381-390 (DOI: http://dx.doi.org/10.4314/afst.v7i1.1).

Brodsky, B.E., Darkhovsky, B.S.,(1993) *Nonparametric Methods in Changepoint Problems*. Kluwer, Dordrecht.

Brown, R. L., Durbin, J. , Evans, J. M. (1975) Techniques for testing the constancy of regression relationships over time (with discussion). *J. R. Statist. Soc. A* 138, 149-63.

Chen, J. and Gupta, A.K. (2011) *Parametric statistical change point analysis: with applications to genetics, medicine, and finance.*, Springer.

Chernoff,H., (1964) Zacks, S. Estimating the current mean distribution which is subjected to change in time, *Ann. Math. Statist*, 35:999-1018.

Chen, C. and Liu, L. M. (1993) Forecasting time series with outliers. *Journal of Forecasting*, 12, 13-35.

Csorgo, M., Horvath, L. (1987) Nonparametric tests for the changepoint problem, *Journal of Statistical Planning and Inference*, 17, 1-9.

Csorgo, M., Horvath, L., (1997) *Limit Theorems in Change-point Analysis*. Wiley, New York.

Deutsch,S. J., Richards, J. E. and Swain, J. J. (1990) Effects of a single outlier on ARMA identification. *Communications in Statistics - Theory and Methods*, 19, 2207-27.

Gelfand, A.E., S.E. Hills, (1990) A. Racine-Poon and A.F.M. Smith, Illustration of Bayesian inference in normal data models using Gibbs sampling, *J. Amer. Statist. Assoc.*, 85, 972-985.

Gelfand, A.E. and Smith, A.F.M., (1990) Sampling-based approaches to calculate marginal densities, J. Amer.Statist. Assoc., 85 398-409.

Gupta, P. K., Singh, A. K. (2017). Classical and Bayesian estimation of Weibull distribution in presence of outliers. Cogent Mathematics,Volume 4, Issue 1,

Jandhyala, V., Fotopoulos, S., MacNeill, I. and Liu, P., (2013) Inference for single and multiple change-points in time series, *J. Time Series Anal.* 34, no. 4, 423-446. (https://doi.org/10.1111/jtsa.12035).

Kander,A. and Zacks, S. (1966) Test procedure for possible change in parameters of statistical distributions occuring at unknown time point, *Ann. Math. Statist*, 37: 1196-1210.

Kim, D., (1991) A Bayesian significance test of the stationarity of regression parametres, *Biometrika*: 78: 667–675, .

Ledolter, J.,(1989) The effect of additive outliers on the forecasts from ARMA models *International Journal of Forecasting*, 5, 231-40.

Liu, Y., Zou, C., Zhang, R., (2008)*Empirical likelihood ratio test for a change-point in linear regression model*. Communications in Statistics-Theory and Methods, 37(16), 2551-2563 (https://doi.org/10.1080/03610920802040373).

Nasiri, P., Pazira, H., (2010) Bayesian Approach on the Generalized Exponential Distribution in the Presence of Outliers, *Journal of Statistical Theory and Practice*, 4:3, 453-475 (https://doi.org/10.1080/15598608.2010.10411997).

Page, E. S. (1954) Continuous inspection schemes. *Biometrika* 41, 100-115.

Page E. S. (1955) A test for change in a parameter occurring at an unknown point. *Biometrika*: 42, 523-527.

Pena, D., (1990) Influential observations in time series. *Journal of Business and Economic Statistics* 8, 235-41.

Slama, A., (2014a) A Bayesian significance test of change for correlated observations, *Discussiones Mathematicae, Probability and Statistics*, 34, 51-63 (doi:10.7151/dmps.1169).

Slama, A.,(2014b) A Bayesian significance test of change in the presence of single outlier, *Afrika statistika*, Volume 9(1), pp 685-694 (ttp://dx.doi.org/10.16929/as/2014.685.62).

Slama, A., Saggou, H., (2017) A bayesian analysis of a change in the parameters of autoregressive time series. *Communications in Statistics-Simulation and Computation*, Volume 46, Issue 9, Pages: 7008-7021 (https://doi.org/10.1080/03610918.2016.1222423).

Verdinelli, I., and Wasserman, L.,(1991) Bayesian analysis of outlier problems using the Gibbs sampler. *Statistics and Computing* 1, 105-117.