A retrial queueing-inventory system with service option on arrival and Multiple vacations

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Abstract. New models of a retrial queueing inventory system with finite buffer and two types of services (say, Type-1 and Type-2) are proposed. Here we discuss a type 1 service, in which a customer enters in the system and receives the service only. In the type-2 service, a customer gets the service and the level of inventory is reduced to one (demand an item). Some important system performance measures in the steady state are derived and long-run total expected cost rate of the proposed model is developed. The results of the numerical examples are shown.

Key words: Continuous review inventory system, Positive lead time, Retrial customers, Multiple vacations, Two types of service and Impatient customers.

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Résumé (French) De nouveaux modèles de système de file d’attente relatifs aux inventaire avec mémoire tampon finie et deux types de services (par exemple, type 1 et type 2) sont proposés. Nous discutons un premier service de type 1 dans lequel un client entre dans le système et ne reçoit que le service. Dans le service de type 2, un client reçoit le service et ne peut demander qu’un article au plus. Certaines mesures importantes de performance du système en régime permanent sont données ainsi que le taux de coût total prévu à long terme. Des résultats d’examles numériques sont fournis.

1. Introduction

Queueing Inventory System (QIS) server vacation model is the most important in inventory management models. In inventory models, the concept of server vacation has been investigated by Daniel and Ramanarayanan (1987, 1988). Narayanan et al. (2008) analyzed Inventory System (IS) of stock out period when the server is on vacation period. Krishnamoorthy et al. (2010) developed a production IS with server vacation and service based on phase type distribution. Sivakumar (2011), Jeyaraman et al. (2012) and Padmavathi et al. (2015) have recently analysed the concept of server vacation models.

The inventory management system also plays an important role in many real life situations e.g. in manufacture, warehouse, supply chains and spare part allocation, etc. The mathematical study of QIS with service facility was initiated by Berman et al. (1993). Infinite capacity QIS with service facility was addressed by Berman and Kim (1999). Krishnamoorthy et al. (2011), Anbazhagan et al. (2014), Berman and Sapna (2000, 2001) have given more references of IS with service facility.

Recently, considerable attention has been given on inventory problems to study Retrial Queuing Inventory System (RQIS) by some of the researchers in different frame work. Artalejo et al. (2006) was the first worked on retrial demands in an IS. As a related work, Ushakumari (2006) considered RQIS with classical retrial policy. Yadavalli et al. (2018), Jeganathan et al. (2016, 2013) have studied RQIS in details. In many real life situations (e.g. call centre and Barber shop setting), the customers impatience (amplified by large customers loads) leads naturally to large number of reneging. Ignoring the presence of reneging can lead to inappropriate sizing of the system. Some of the researchers have taken into account of the customer’s impatient behavior in their works and poor staffing allocation. The present study points out some of the references Paul Manual et al. (2006), Yadavalli et al. (2015) and Rajkumar (2014).

Motivated by such practical situations, this paper considers RQIS, single server served at the two types of service, multiple vacations and impatient customers. Customers are arrived in a single server counter according to a Poisson process. The inventory is replenished according to \((s, Q)\) policy and the replenishing times
are assumed to be exponentially distributed. When the inventory level is zero, server goes to a vacation. If the server finds the empty stock at the end of a vacation, he immediately takes another vacation. Otherwise, the server starts his service on any arriving customers. When the waiting hall is full, any new arriving customer enters into the orbit of finite size and tries at a future time, which is exponentially distributed. The joint probability distribution of the number of customers in the orbit, the number of customers in the waiting hall, the server status and the inventory level is obtained for the steady state case. Various system performance measures in the steady state are derived and the long-run total expected cost rate is calculated.

The remainder of this paper is organized as follows. In section 2, the description of the mathematical model and notations are presented. Analysis of the model, the state equations governing the model are constructed in section 3. In section 4, various measures of system performance in steady state are derived. The total expected cost rate is calculated and cost function is carried out numerically in section 5. Last section gives the conclusion of the paper.

2. Model description

The behavior of the RQIS can be described as follows:

- Continuous time RQIS with maximum capacity of \( S \) units.
- As the \((s,Q)\) replenishment policy, when on-hand inventory level drops to a prefixed level, say \( s(>0) \) an order for \( Q(=S-s+s+1) \) units is placed. The lead time is exponentially distributed with the rate \( \beta \).
- Customers arrive in the system one by one according to a Poisson stream with arrival rate \( \lambda(>0) \). The waiting room of the demands and the storage size are assumed to be \( N \).
- Each arriving customer may choose either type 1 service with probability \( p_1 \) or type 2 service with probability \( p_2 \). The service time for both types are following exponential distribution with rate \( p_i\mu(>0), i=1,2 \).
- The completion of type-1 service, one customer receiving service only and type-2 service, one customer receiving service and reduces the inventory by one item.
- When the inventory level is zero, the server takes a vacation of random length. The vacation time also follows an exponential distribution with rate \( \gamma \).
- On returning from vacation the server finds empty stock, he immediately commences another vacation otherwise the server resumed to a regular service any arriving customers.
- Since an arriving customer will enter the orbit only if there are atmost \( N \) customers in the waiting hall, the maximum number of customers located in orbit is equal to \( M \). Customers located in the orbit retry to enter the waiting hall with retrial rate \( \theta \). We assume constant retrial policy for the retrial customers.
- When the orbit is full, arriving primary customer is considered to be lost.
- If the waiting hall customer finds the server is on vacation, the customer becomes independently impatient and leaves the system with rate \( \eta(>0) \), which is exponentially distributed.

Notations

0 : Zero matrix.
I : Identity matrix.
e : A column vector of 1’s of appropriate dimension.
$[A]_{ij}$ : Entry at $(i,j)$th position of a matrix $A$.
$\delta_{ij}$ : \[
\begin{cases}
1, & \text{if } i = j, \\
0, & \text{otherwise}.
\end{cases}
\]
$\bar{\delta}_{ij}$ : 

$Z(t)$ : \[
\begin{cases}
0, & \text{if server is on vacation at time } t. \\
1, & \text{if server is not on vacation at time } t.
\end{cases}
\]

3. Analysis

By the description of the model, we know that at time $t$, the state of the system considered in this paper can be described by the Markov process $\{(X(t), Y(t), Z(t), L(t)), t \geq 0\}$, where $X(t)$ denote the number of customers in the orbit at time $t$, $Y(t)$ is the number of customers in the waiting hall at time $t$, $Z(t)$ denotes the server status at time $t$ and $L(t)$ represents the inventory level of the commodity at time $t$ with state space given by $E$. Where

$E = \{(i,j,0,l) : i = 0,1,2,\ldots,M, j = 0,1,2,\ldots,N, l = 0, Q\} \cup \{(i,j,1,l) : i = 0,1,2,\ldots,M, j = 0,1,2,\ldots,N, l = 1,2,\ldots,S\}$

To determine the infinitesimal generator

$A = (a((i,j,k,l),(i',j',k',l')))$, \quad $(i,j,k,l),(i',j',k',l') \in E$

of this process we use the following arguments :

- An arrival of a primary customer makes a transition from $(i,j,1,l)$ to $(i,j+1,1,l)$ with intensity of transition $\lambda, i = 0,1,2,\ldots,M, j = 0,1,2,\ldots,N-1, l = 1,2,\ldots,S$

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- An arrival of a primary customer makes a transition from $(i,N,1,l)$ to $(i+1,N,1,l)$ with intensity of transition $\lambda, i = 0,1,2,\ldots,M-1, l = 1,2,\ldots,S$

- An arrival of a primary customer makes a transition from $(i,N,0,l)$ to $(i+1,N,0,l)$ with intensity of transition $\lambda, i = 0,1,2,\ldots,M-1, l = 0,0, Q$

- The completion of type-1 service makes one customer leaves the system. Thus a transition takes place from $(i,j,1,l)$ to $(i,j-1,1,l)$ with intensity of transition $\mu, i = 0,1,2,\ldots,M, j = 1,2,\ldots,N, l = 1,2,3,\ldots,S$

The completion of type-2 service makes one customer leaves the system and decreases the inventory level by one. Thus a transition takes place from $(i, j, 1, l)$ to $(i, j - 1, 1, l - 1)$ with intensity of transition $p_2\mu$, $i = 0, 1, 2, \ldots, M$, $j = 1, 2, \ldots, N$, $l = 2, 3, \ldots, S$

The completion of type-2 service makes one customer leaves the system and decreases the inventory level by one. Thus a transition takes place from $(i, j, 1, l)$ to $(i, j - 1, 0, 0)$ with intensity of transition $p_2\mu$, $i = 0, 1, 2, \ldots, M$, $j = 1, 2, \ldots, N$

A retrial requests of an orbiting customer makes a transition from $(i, j, k, l)$ to $(i - 1, j + 1, k, l)$ with intensity of transition $\theta$, $i = 1, 2, \ldots, M$, $j = 0, 1, 2, \ldots, N$, $k = 0, 1$, $l = 0, 1, 2, \ldots, S$

A reneging of a waiting hall customer makes a transition from $(i, j, 0, l)$ to $(i, j - 1, 0, l)$ with intensity of transition $\eta$, $i = 0, 1, 2, \ldots, M$, $j = 1, 2, \ldots, N$, $l = 0, Q$

When the vacation end, if the inventory level is positive, the server starts his service, otherwise the server takes another vacation. The intensity of transition from $(i, j, 0, l)$ to $(i, j, 1, l)$ is given by $\gamma$, $i = 0, 1, 2, \ldots, M$, $j = 0, 1, 2, \ldots, N$, $l = Q$

A transition from $(i, j, k, l)$ to $(i, j, k, l + Q)$ for $i = 0, 1, \ldots, M$, $j = 0, 1, \ldots, N$, $k = 0, l = 0$ or for $i = 0, 1, \ldots, M$, $j = 0, 1, \ldots, N$, $k = 1, l = 1, 2, \ldots, s$ with intensity $\beta$ when a replenishment occurs.

We observe that no transition other than the above is possible.

Finally, the value of $a((i, j, k, l), (i, j, k, l))$ is obtained by

$$a((i, j, k, l), (i, j, k, l)) = - \sum_{i'} \sum_{j'} \sum_{k'} \sum_{l'} a((i, j, k, l), (i', j', k', l')) \cdot a((i', j', k', l'), (i, j, k, l)).$$

By ordering states lexicographically, the infinitesimal generator $A = (a((i, j, k, l), (i', j', k', l')))$, $(i, j, k, l), (i', j', k', l') \in E$ can be conveniently expressed in a block partitioned matrix with entries
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\[ [A]_{ij} = \begin{cases} 
  A_2, & j = i, \quad i = M \\
  A_1, & j = i, \quad i = 1, 2, \ldots, M - 1 \\
  A_0, & j = i, \quad i = 0 \\
  B, & j = i + 1, \quad i = 0, 1, 2, \ldots, M - 1 \\
  C, & j = i - 1, \quad i = 1, 2, \ldots, M \\
  0, & \text{otherwise}
\end{cases} \]

Where

\[ [A_0]_{ij} = \begin{cases} 
  D, & j = i, \quad i = 0 \\
  E, & j = i, \quad i = 1, 2, \ldots, N \\
  F, & j = i + 1, \quad i = 0, 1, 2, \ldots, N - 1 \\
  G, & j = i - 1, \quad i = 1, 2, \ldots, N \\
  0, & \text{otherwise}
\end{cases} \]

\[
D = \frac{1}{1} \begin{pmatrix} D_{11} & 0 \\ D_{01} & D_{00} \end{pmatrix}, \quad 
E = \frac{1}{1} \begin{pmatrix} E_{11} & 0 \\ E_{01} & E_{00} \end{pmatrix}, \quad 
F = \frac{1}{1} \begin{pmatrix} F_{11} & 0 \\ 0 & F_{00} \end{pmatrix},
\]

\[
G = \frac{1}{1} \begin{pmatrix} G_{11} & 0 \\ 0 & G_{00} \end{pmatrix}
\]

\[
[D_{11}]_{kl} = \begin{cases} 
  \beta, & l = k + Q, \quad k = s, s - 1, \ldots, 1 \\
  -\lambda, & l = k, \quad l = S, S - 1, \ldots, s + 1 \\
  -(\lambda + \beta), & l = k, \quad k = s, s - 1, \ldots, 1 \\
  0, & \text{otherwise}
\end{cases} \]

\[
[D_{01}]_{kl} = \begin{cases} 
  \gamma, & l = k, \quad k = Q \\
  0, & \text{otherwise}
\end{cases} \]
\[ [D_{00}]_{kl} = \begin{cases} 
\beta, & l = k + Q, \quad k = 0 \\
-(\lambda + \beta), & l = k, \quad l = 0 \\
-(\lambda + \gamma), & l = k, \quad k = Q \\
0, & \text{otherwise}
\end{cases} \]

\[ [E_{11}]_{kl} = \begin{cases} 
\beta, & l = k + Q, \quad k = s, s - 1, \ldots, 1 \\
-(\lambda + p_1 \mu + p_2 \mu), & l = k, \quad l = S, S - 1, \ldots, s + 1 \\
-(\lambda + \beta + p_1 \mu + p_2 \mu), & l = k, \quad k = s, s - 1, \ldots, 1 \\
0, & \text{otherwise}
\end{cases} \]

\[ [E_{00}]_{kl} = \begin{cases} 
\beta, & l = k + Q, \quad k = 0 \\
-(\lambda + \beta + \eta), & l = k, \quad l = 0 \\
-(\lambda + \gamma + \eta), & l = k, \quad k = Q \\
0, & \text{otherwise}
\end{cases} \]

\[ [F_{11}]_{kl} = \begin{cases} 
p_1 \mu, & l = k, \quad k = S, S - 1, \ldots, 1 \\
p_2 \mu, & l = k - 1, \quad k = S, S - 1, \ldots, 1 \\
0, & \text{otherwise}
\end{cases} \]

\[ [F_{00}]_{kl} = \begin{cases} 
\eta, & l = k, \quad k = 0, Q \\
0, & \text{otherwise}
\end{cases} \]

\[ [G_{11}]_{kl} = \begin{cases} 
\lambda, & l = k, \quad k = S, S - 1, \ldots, 1 \\
0, & \text{otherwise}
\end{cases} \]

\[ [G_{00}]_{kl} = \begin{cases} 
\lambda, & l = k, \quad k = 0, Q \\
0, & \text{otherwise}
\end{cases} \]
\[ \begin{align*}
[A_1]_{ij} & = \begin{cases} 
D_1, & j = i, \quad i = 0 \\
E_1, & j = i, \quad i = 1, 2, \ldots, N \\
F_1, & j = i + 1, \quad i = 0, 1, 2, \ldots, N - 1 \\
G_1, & j = i - 1, \quad i = 1, 2, \ldots, N \\
0, & \text{otherwise}
\end{cases} \\
D_1 & = \begin{pmatrix} 1 & 0 \\
\begin{pmatrix} D_{11}^{(1)} & 0 \\
0 & D_{01}^{(1)} \\
D_{00}^{(1)} \end{pmatrix} \end{pmatrix}, \\
E_1 & = \begin{pmatrix} 1 & 0 \\
\begin{pmatrix} E_{11}^{(1)} & 0 \\
0 & E_{01}^{(1)} \\
E_{00}^{(1)} \end{pmatrix} \end{pmatrix}, \\
F_1 & = \begin{pmatrix} 1 & 0 \\
\begin{pmatrix} F_{11}^{(1)} & 0 \\
0 & F_{01}^{(1)} \\
F_{00}^{(1)} \end{pmatrix} \end{pmatrix}, \\
G_1 & = \begin{pmatrix} 1 & 0 \\
\begin{pmatrix} G_{11}^{(1)} & 0 \\
0 & G_{01}^{(1)} \\
G_{00}^{(1)} \end{pmatrix} \end{pmatrix}
\end{align*} \]

\[ \begin{align*}
[D_{11}^{(1)}]_{kl} & = \begin{cases} 
\beta, & l = k + Q, \quad k = s, s - 1, \ldots, 1 \\
-(\lambda + \theta), & l = k, \quad l = S, S - 1, \ldots, s + 1 \\
-(\lambda + \beta + \theta), & l = k, \quad k = s, s - 1, \ldots, 1 \\
0, & \text{otherwise}
\end{cases} \\
[D_{00}^{(1)}]_{kl} & = \begin{cases} 
\beta, & l = k + Q, \quad k = 0 \\
-(\lambda + \beta + \theta), & l = k, \quad l = 0 \\
-(\lambda + \gamma + \theta), & l = k, \quad k = Q \\
0, & \text{otherwise}
\end{cases}
\end{align*} \]

\[ \begin{align*}
[A_2]_{ij} & = \begin{cases} 
D_2, & j = i, \quad i = 0 \\
E_2, & j = i, \quad i = 1, 2, \ldots, N - 1 \\
E_3, & j = i, \quad i = N \\
F_2, & j = i + 1, \quad i = 0, 1, 2, \ldots, N - 1 \\
G_2, & j = i - 1, \quad i = 1, 2, \ldots, N \\
0, & \text{otherwise}
\end{cases}
\end{align*} \]
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\[
\begin{align*}
D_2 &= \begin{pmatrix} 1 & 0 \\ 0 & D_{01}^{(2)} \end{pmatrix}, \\
E_2 &= \begin{pmatrix} 1 & 0 \\ 0 & E_{01}^{(2)} \end{pmatrix}, \\
F_2 &= \begin{pmatrix} 1 & 0 \\ 0 & F_{01}^{(2)} \end{pmatrix}, \\
E_3 &= \begin{pmatrix} 1 & 0 \\ 0 & E_{01}^{(3)} \end{pmatrix}.
\end{align*}
\]

\[
[\mathbf{E}_{11}^{(3)}]_{kl} = \begin{cases} 
\beta, & l = k + Q, \quad k = s, s-1, \ldots, 1 \\
-(p_1 \mu + p_2 \mu), & l = k, \quad l = S, S-1, \ldots, s+1 \\
-(\beta + p_1 \mu + p_2 \mu), & l = k, \quad k = s, s-1, \ldots, 1 \\
0, & \text{otherwise}
\end{cases}
\]

\[
[\mathbf{E}_{00}^{(3)}]_{kl} = \begin{cases} 
\beta, & l = k + Q, \quad k = 0 \\
-(\beta + \eta), & l = k, \quad l = 0 \\
-(\gamma + \eta), & l = k, \quad k = Q \\
0, & \text{otherwise}
\end{cases}
\]

\[
D_{01} = D_{01}^{(1)} = D_{01}^{(2)}, \quad E_{01} = E_{01}^{(1)} = E_{01}^{(2)} = E_{01}^{(3)}, \quad E_{11} = E_{11}^{(1)} = E_{11}^{(2)}, \quad E_{00} = E_{00}^{(1)} = E_{00}^{(2)}.
\]

\[
[\mathbf{B}]_{ij} = \begin{cases} 
B_1, & j = i, \quad i = N \\
0, & \text{otherwise}
\end{cases}
\]

\[
B_1 = \begin{pmatrix} 1 & 0 \\ 0 & B_{00} \end{pmatrix}.
\]

\[
[B_{11}]_{kl} = \begin{cases} 
\lambda, & l = k, \quad k = S, S-1, S-2, \ldots, 1 \\
0, & \text{otherwise}
\end{cases}
\]

\[
[B_{00}]_{kl} = \begin{cases} 
\lambda, & l = k, \quad k = 0, Q \\
0, & \text{otherwise}
\end{cases}
\]

$[C]_{ij} = \begin{cases} C_1, & j = i, \\ 0, & \text{otherwise} \end{cases} \quad \quad \quad \quad \quad i = N

C_1 = \begin{pmatrix} 1 & 0 \\ 0 & C_{00} \end{pmatrix}.

[C_{11}]_{kl} = \begin{cases} \theta, & l = k, \\ 0, & \text{otherwise} \end{cases} \quad \quad \quad \quad \quad k = S, S-1, S-2, \ldots, 1

[C_{00}]_{kl} = \begin{cases} \theta, & l = k, \\ 0, & \text{otherwise} \end{cases} \quad \quad \quad \quad \quad k = 0, Q

3.1. Steady State Analysis

It can be seen from the structure of $A$ that the homogeneous Markov process $\{(X(t), Y(t), Z(t), L(t)), \ t \geq 0\}$ on the finite state space $E$ is irreducible. Hence the limiting distribution

$\pi(i_1, i_2, i_3, i_4) = \lim_{t \to \infty} \text{pr}\{X(t) = i_1, Y(t) = i_2, Z(t) = i_3, L(t) = i_4 | X(0), Y(0), Z(0), L(0)\},$

exists. Let

$\Pi = (\Pi^{(0)}, \Pi^{(1)}, \Pi^{(2)}, \ldots, \Pi^{(M)})$

we partition the vector, $\Pi^{(i_1)}$ into as follows:

$\Pi^{(i_1)} = (\Pi^{(i_1,0)}, \Pi^{(i_1,1)}, \Pi^{(i_1,2)}, \ldots, \Pi^{(i_1,N)}), \quad i_1 = 0, 1, 2, \ldots, M$

which is partitioned as follows:

$\Pi^{(i_1,i_2)} = (\pi^{(i_1,i_2,0,0)}, \pi^{(i_1,i_2,0,Q)}, \pi^{(i_1,i_2,1,1)}, \pi^{(i_1,i_2,1,2)}, \ldots, \pi^{(i_1,i_2,1,S)})$

for $i_1 = 0, 1, 2, \ldots, M, \quad i_2 = 0, 1, 2, \ldots, N$

Then the limiting probability, $\Pi$ satisfies

$\Pi A = 0, \quad \Pi e = 1 \quad (1)$

From the structure of $A$, it is a finite QBD matrix, therefore its steady state vector $\Pi$ can be computed by using the following algorithm described by Gaver et al. (1984).

Algorithm :
1. Determine recursively the matrices

\[ F_0 = A_0 \]
\[ F_i = A_1 + B\left(-F_{i-1}\right)C, \quad i = 1, 2, \ldots, M - 1, \]
\[ F_M = A_2 + B\left(-F_{M-1}\right)C. \]

2. Compute recursively the vectors \( \Pi^{(i)} \) using

\[ \Pi^{(i)} = \Pi^{(i+1)} B\left(-F_{i-1}\right), \quad i = 0, 1, 2, \ldots, M - 1 \]

3. Solve the system of equations

\[ \Pi^{(M)} F_M = 0 \text{ and } \sum_{i=0}^{M} \Pi^{(i)} e = 1. \]

From the system of equations \( \Pi^{(M)} F_M = 0 \), vector \( \Pi^{(M)} \) could be determined uniquely, up to a multiplicative constant. This constant is decided by \( \Pi^{(i)} = \Pi^{(i+1)} B\left(-F_{i-1}\right), \quad i = 0, 1, 2, \ldots, M - 1 \) and \( \sum_{i=0}^{M} \Pi^{(i)} e = 1 \).

4. System Performance Measures

In this section, we establish some performance indices, which are of great interest in order to determine the performance and efficiency of the system and are given below:

4.1. Expected inventory level

Let \( \rho_I \) denote the mean inventory level in the steady state. Then

\[ \rho_I = \sum_{i_1=0}^{M} \sum_{i_2=0}^{N} \left[ \sum_{i_4=1}^{S} i_4 \pi^{(i_1,i_2,1,i_4)} + Q \pi^{(i_1,i_2,0,Q)} \right] \]

4.2. Expected reorder rate

Let \( \rho_R \) denote the expected reorder rate in the steady state. Then

\[ \rho_R = \sum_{i_1=0}^{M} \sum_{i_2=1}^{N} p_{2\mu} \left[ \pi^{(i_1,i_2,1,s+1)} \right] \]
4.3. Expected number of demands in the waiting hall

Let $\rho_W$ denote the expected number of demands in the waiting hall in the steady state. Then

$$\rho_W = \sum_{i_1=0}^{M} \sum_{i_2=1}^{N} \sum_{i_4=1}^{S} i_2 \left[ \pi^{(i_1,i_2,1,i_4)} + \pi^{(i_1,i_2,0,0)} + \pi^{(i_1,i_2,0,Q)} \right]$$

4.4. Expected number of demands in the orbit

Let $\rho_O$ denote the expected number of demands in the orbit in the steady state. Then

$$\rho_O = \sum_{i_1=1}^{M} \sum_{i_2=0}^{N} \sum_{i_4=1}^{S} i_1 \left[ \pi^{(i_1,i_2,1,i_4)} + \pi^{(i_1,i_2,0,0)} + \pi^{(i_1,i_2,0,Q)} \right]$$

4.5. Overall rate of retrials

Let $\rho_{OR}$ denote the overall rate of retrials in the steady state. Then

$$\rho_{OR} = \sum_{i_1=1}^{M} \sum_{i_2=0}^{N} \sum_{i_4=1}^{S} \theta \left[ \pi^{(i_1,i_2,1,i_4)} + \pi^{(i_1,i_2,0,0)} + \pi^{(i_1,i_2,0,Q)} \right]$$

4.6. The successful retrial rate

Let $\rho_{SR}$ denote the successful retrial rate in the steady state. Then

$$\rho_{SR} = \sum_{i_1=1}^{M} \sum_{i_4=1}^{S} \theta \left[ \pi^{(i_1,0,1,i_4)} + \pi^{(i_1,0,0,0)} + \pi^{(i_1,0,0,Q)} \right]$$

4.7. The fraction of successful rate of retrial

Let $\rho_{FR}$ denote the successful retrial rate in the steady state. Then

$$\rho_{FR} = \frac{\rho_{SR}}{\rho_{OR}}$$

4.8. Fraction of time server is on vacation

Let $\rho_{FV}$ denote the server is on vacation in the steady state. Then

$$\rho_{FV} = \sum_{i_1=0}^{M} \sum_{i_2=0}^{N} \left[ \pi^{(i_1,i_2,0,0)} + \pi^{(i_1,i_2,0,Q)} \right]$$
4.9. Expected blocking rate

Let $\rho_B$ denote the expected blocking rate in the steady state. Then

$$\rho_B = \sum_{i_1=1}^{M-1} \sum_{i_4=0}^{S} \lambda \left[ \pi(i_1,N,1,i_4) + \pi(i_1,N,0,0) + \pi(i_1,N,0,Q) \right]$$

4.10. Expected reneging rate

Let $\rho_{Rg}$ denote the expected reneging rate in the steady state. Then

$$\rho_{Rg} = \sum_{i_2=0}^{N} \sum_{i_2=1}^{N} \eta \left[ \pi(i_1,i_2,0,0) + \pi(i_1,i_2,0,Q) \right]$$

5. Cost Analysis

The expected total cost per unit time (expected total cost rate) in the steady state for this model is defined to be

- $c_h$: the inventory holding cost per unit item per unit time
- $c_s$: the inventory setup cost per unit item per unit time
- $c_b$: cost per blocking customer
- $c_w$: Waiting cost of a customer in the waiting hall per unit time.
- $c_o$: Waiting cost of a customer in the orbit per unit time.
- $c_r$: reneging cost per customer per unit time.

The long run total expected cost rate is given by

$$TC(S,s,N,M) = c_h \rho_I + c_s \rho_R + c_b \rho_B + c_w \rho_W + c_o \rho_O + c_r \rho_{Rg}$$

Substituting $\rho$'s into the above equation, we obtain
\[ TC(S, s, N, M) = c_b \sum_{i_1=0}^{M} \sum_{i_2=0}^{N} \left[ \sum_{i=1}^{S} p_2(M-1) \pi_{i_1,i_2}^{(1)} + Q \pi_{i_1,i_2,0}^{(1)} \right] + c_s \]

\[ \sum_{i_1=0}^{M} \sum_{i_2=0}^{N} p_2 \mu \left[ \pi_{i_1,i_2,1}^{(1)} \right] + \left[ \pi_{i_1,i_2,0}^{(1)} + \pi_{i_1,i_2,0}^{(2)} + \pi_{i_1,i_2,0}^{(3)} \right] \]

\[ + c_w \sum_{i_1=0}^{M} \sum_{i_2=1}^{N} \sum_{i_4=1}^{S} \left[ \pi_{i_1,i_2,1,i_4}^{(1)} + \pi_{i_1,i_2,0,i_4}^{(2)} + \pi_{i_1,i_2,0,i_4}^{(3)} \right] + c_r \sum_{i_1=0}^{M} \sum_{i_2=0}^{N} \sum_{i_4=1}^{S} \left[ \pi_{i_1,i_2,0,i_4}^{(1)} + \pi_{i_1,i_2,0,i_4}^{(2)} + \pi_{i_1,i_2,0,i_4}^{(3)} \right] \]

Due to the complex form of the limiting distribution, it is difficult to discuss the qualitative behavior of the cost function \( TC(S, s, N, M) \) analytically. Hence, a detailed computational study of the total expected cost rate function is carried out in the next section.

### 5.1. Numerical Examples

In this section, we investigate some numerical examples that reveal the possible convexity of the total expected cost rate. We explore the behavior of the cost function by considering it as function of any two variables by fixing the others are constants.

Table 1 and 2 give the total expected cost rate as a function of \( TC(S, s, 9, 4) \) and \( TC(S, 4, N, 4) \). All costs and other parameters are assigned fixed values which are indicated in each table. The value that is shown bold is the least among the values in that column and the value that is shown underlined is the least in that row. It may be observed that, these values in each Table exhibit a (possibly) local minimum of the function of the two variables.

**Example 1:** In this example, we investigate the effects of some parameters on the performance measures of the system and show an optimum value in terms of the cost function \( TC \). From table 3-6, the variations of \( \rho_l, \rho_R, \rho_W, \rho_O, \rho_B, \rho_{Rg} \) and \( TC \) are presented for various parameters \( \lambda = 4, \theta = 1.2, \beta = 1.5, \mu = 1.6, \eta = 0.02, \gamma = 0.58, p_1 = 0.4 \) and \( p_2 = 0.6 \). We observe the following from Table 3-6.

1. The system performance measures \( \rho_l, \rho_R, \rho_W, \rho_O \). \( TC \) increase and \( \rho_B \) decreases, when arrival rate \( \lambda \) increases.
2. The system performance measures \( \rho_l, \rho_R, \rho_B, \rho_{Rg} \). \( TC \) increase and \( \rho_W, \rho_O \) decrease, when service rate \( \mu \) increases.
3. The system performance measures \( \rho_l, \rho_W, \rho_B, \rho_{Rg} \). \( TC \) increase and \( \rho_O \) decreases, when retrial rate \( \theta \) increases.
4. The system performance measures \( \rho_{Rg}, \rho_{Rg} \). \( TC \) increase and \( \rho_W \) decreases, when renaging rate \( \eta \) increases.
Table 1. Total expected cost rate as a function of $S$ and $s$

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<tr>
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Table 2. Total expected cost rate as a function of $S$ and $N$

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5. The system performance measures $\rho_R$ increases and $\rho_I$, $\rho_W$, $\rho_O$, $\rho_{Rg}$, $TC$ decrease, when reorder rate $\beta$ increases.

Example 2: In this example, we study the impact of $c_h$, $c_s$, $c_w$, $c_b$, $c_o$ and $c_r$ on the optimal values $(S^*, s^*)$ and the corresponding total expected cost rate $TC^*$. By fixing the parameter values as $\lambda = 4$, $\theta = 1.2$, $p_1 = 0.4$, $p_2 = 0.6$, $\beta = 1.5$, $\gamma = 0.58$, $\mu = 1.6$, $\eta = 0.02$, $c_h = 0.3$, $c_s = 13$, $c_w = 0.05$, $c_b = 3.02$, $c_o = 0.2$, $c_r = 1$, $s = 4$, $M = 4$. We observe the following from table 8 to 13.

1. The Total expected cost rate increases when $c_h$, $c_s$, $c_w$, $c_b$, $c_o$ and $c_r$ increase.
2. If $c_s$ increases, then $S^*$ monotonically increases. If $c_h$ increases, then $S^*$ monotonically decreases.
Table 3. Effects of $\lambda$ and $\mu$ on some performance measures and total cost.

<table>
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<tr>
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<th>$\mu$</th>
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<th>$\rho_W$</th>
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Table 4. Effects of $\lambda$ and $\beta$ on some performance measures and total cost.

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Table 5. Effects of $\lambda$ and $\eta$ on some performance measures and total cost.

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Table 6. Effects of $\theta$ and $\mu$ on some performance measures and total cost.

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Table 7. Effect of $c_h$ and $c_s$ on optimal values

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6. Conclusion and Perspectives

We have proposed a new model of stochastic inventory system with finite buffer. We discussed in this paper two different types of services. In type-1 service we assumed, a customer enters into the system and receiving the service only (not demand an item). In type-2 service we assumed, a customer getting the service and the level of inventory is reduced by one (demand an item). The model is analyzed within the framework of Markov processes. Joint probability distribution of the number of customers in the orbit, the number of customers in the waiting hall, the server status and the inventory level is obtained for the steady state case. Various system performance measures in the steady state are derived and the long-run total expected cost rate is calculated. By assuming a appropriate cost structure on the queueing-inventory system, we have presented extensive numerical illustrations to show the effect of change of values for constants on the total expected cost rate. The authors are working in the direction of MAP (Markovian arrival process) arrival for the customers and service times that follow PH-distributions.

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References


Narayanan V. C., Deepak T. G., Krishnamoorthy A. and Krishnakumar B., 2008. On an (s, S) inventory policy with service time, vacation to server and correlated lead time. *Quantitative Technology and Quantitative Management*, 5(2), 129-143.


