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Forecasting Value-at-Risk using Markov Regime-Switching asymmetric *GARCH* model with Stable distribution in the context of the *COVID-19* pandemic

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Abstract. The spread of the coronavirus is putting a strain on financial markets and the resulting stock market volatility is causing huge problems for investors. Volatility in the *U.S.* market has returned to levels not seen since the 2011 sovereign debt crisis. It is already clear that this volatility has had a negative effect on the economy. In this study, we introduce a regime-switching *GJR-GARCH* model with a stable distribution to investigate the predictive power of the *S&P 500* index volatility to *VaR* estimation. The results of *VaR* backtesting at a 5% risk level confirm that the model performs better and is a useful tool for the risk manager and financial regulator.

Key words: Value-at-Risk; Markov-switching; *GJR-GARCH*; stable distribution
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Résumé. La propagation du coronavirus met les marchés financiers à rude épreuve et la volatilité boursière qui en résulte pose d'énormes problèmes aux investisseurs. La volatilité sur le marché américain est revenue à des niveaux jamais vus depuis la crise de la dette souveraine de 2011. Il est déjà clair que cette volatilité a eu un effet négatif sur l'économie. Dans cette étude, nous introduisons un modèle *GJR-GARCH* à changement de régime avec une distribution stable pour étudier le pouvoir prédictif de la volatilité de l'indice *S&P 500* à l'estimation de la *VaR*. Les résultats du backtesting de la *VaR* à un niveau de risque de 5% confirment que le modèle est plus performant et constitue un outil utile pour le gestionnaire de risque et le régulateur financier.

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1. Introduction

Since February 2020, new cases of *COVID-19* reported by the rest of the world have been observed on a clear upward trend, with the increase in infectious cases rapidly overtaking China, the initial epicenter of the *COVID-19* epidemic. The *WHO* officially announced that *COVID-19* was a global pandemic on March 11, 2020, just days after the *WHO* declared the new virus a public health emergency of international concern on January 30, 2020. Since then, governments around the world have imposed historic containment measures to control the spread of the *COVID-19* virus such as closing entertainment facilities, suspending commercial and non-commercial activities such as controlling movement within the country, closing borders with strict controls, and closing all school levels, etc. The spread of the *COVID-19* virus is still ongoing with more than 100 million *COVID-19* cases reported, with a number of deaths approaching 2 million worldwide even though by the end of November 2020, the announcement of the vaccine discovery was made. The consequences of this pandemic are far beyond our imagination. Almost all economic activities have been directly or indirectly limited by the pandemic, which has caused unprecedented damage to the global economy.

Ma *et al.* (2020) examine the impact of the five previous infectious diseases, including *SARS* [2003], *H1N1* [2009], *MERS* [2012], *Ebola* [2014] and *Zika* [2016] in the 21st century to understand the potential effects of *COVID-19* on the global economy and financial markets. The authors find that the stock market is more likely to overreact in the short term, which is consistent with the preliminary presentation by Phan and Narayan (2020) *WHO* argue that the initial overreaction is due to investor fears about the uncertainty of the development of the *COVID-19* virus. Sharif *et al.* (2020) argue that there is a strong negative impact on the *U.S.* stock market in the short term and that the possibility of a spread of optimism among long-term investors cannot be ruled out with respect to the confidence that the government has built up through its effective intervention.

To measure the potential losses of portfolios under adverse market conditions, the financial industry and regulators have developed rules. One of the objectives of these efforts was to provide banks with a framework for determining the capital necessary for their survival in time of economic and financial difficulty. Risk measurement is a necessary condition of risk management: institutions must be able to quantify the amount of risk they face and effectively develop a strategy to mitigate the consequences of extreme adverse events. In particular, financial risk in equity markets is market risk, which represents the potential losses resulting from adverse movements in equity or currency markets, interest rates, etc.

In the literature, Value-at-Risk (*VaR*) is commonly accepted as the standard measure of market risk and indicates the maximum probable loss on a given portfolio, with reference to a specific confidence level and time horizon. An extensive literature exists on *VaR*, including statistical descriptions and reviews according to the following models (Giot and Laurent (2003)). Historically, the literature on *VaR* has evolved along parametric and non-parametric lines. While in the latter case, the probability distribution of future returns is "simulated from the past" in order to estimate the relevant quantile (i.e. *VaR*), the parametric approach is based on fitting a certain family of probability laws to observed historical returns. *VaR* at an $\alpha\%$ level is estimated as the loss that might be exceeded $\alpha\%$ of the time. Like many other models in finance, it is often based on an assumption that losses follow a normal distribution. Extreme losses are greater than and occur much more often than, a normal distribution would predict. For this purpose, *VaR* measurements are sometimes based on a t-distribution or on an *ARCH*/*GARCH* system, with the innovations of normal or t-distribution. Several other distributions or mixtures of distributions have been proposed, but none has received universal acceptance, and it is probable that none ever will. Empirically, it has been found that equity returns are sometimes negatively correlated with changes in volatility: volatility tends to increase after bad news and decrease after good news. This is the leverage effect. Leverage is not easily detectable in stock market indexes and a company's leverage ratio increases when its share price falls. If the company's cash flow is constant, it will increase the volatility of the stock's return. In this case, negative returns today could be expected to lead to greater volatility tomorrow, and vice versa for positive returns. This behavior cannot be captured by standard *GARCH*(1,1) models. Symmetric *VaR* models of the *GARCH* class have difficulties in correctly modeling the tails of the distribution of returns (Giot and Laurent (2003)) because of leverage effects. The use of asymmetric conditional models that contain an asymmetry parameter in the conditional variance equation and the use of asymmetric density functions for the error terms allows leverage to be taken into account in the volatility forecast.

Although these approaches offer an improvement in fit over symmetric models, empirical evidence suggests that the persistence of the conditional variance is likely to be significantly biased upwards.

Initially, Engle (1982) assumed that this was a normal distribution in financial time series to take into account the distribution of the error term. Then, Bollerslev (1987) proposed that the student-t distribution is more appropriate than the normal distribution. Nelson (1991) claimed that Generalized Error Distribution (*GED*) is more appropriate since it considers the fat tails of financial data. Wilhelmsson (2006) confirmed that the error distribution is important in improving the results when using leptokurtic error distributions instead of the normal distribution. Thus, the symmetric versions of the Gaussian, Student-t, and *GED* distributions are applied in this study. These distributions fail to describe the empirical evidence in financial markets. A possible alternative is to introduce stable laws (Lévy (1925)) what Mandelbrot (1963) and Fama (1965) proposed in the 1960s. Apart from that, they can account for heavy tails and the behavior asymmetrical, on the other hand, dependent on four parameters, the stable laws are more flexible than normal, Student-t, *GED* laws to adjust empirical data in the estimation. The family of stable distributions or α-stable distribution replaces the generally used fat-tailed distribution. Calzolari *et al.* (2014) proposed it.

A Markov regime-switching (*MS*) approach solves this problem by endogenizing changes in the data generation process. In order to allow *GARCH*-type heteroskedasticity within the regime, Gray (1996) extended Hamilton's *MS* model to the *MS-GARCH* model. This model was then modified by Klaassen (2002). Marcucci (2005) compares a set of *GARCH*, *EGARCH*, and *GJR-GARCH* models in an *MS-GARCH* model (Gaussian, Student's t and generalized error distribution for innovations) in terms of their ability to predict $S\&P$ 100 volatilities. Ané and Ureche-Rangau (2006) extend the regime-switching model developed by Gray (1996) to an Asymmetric Power (*AP*) *GARCH* model and evaluate its performance on four Asian stock market indices. Sajjad *et al.* (2008) apply an asymmetric Markov regime-switching *GARCH* (*MRS-GARCH*) model to estimate Value-at-Risk for long and short positions of the *FTSE* 100 and *S&P 500* index. The study shows that *MRS-GARCH* under a Student-t distribution for the innovations outperforms other models in estimating the *VaR* for both long and short positions of the *FTSE* returns data. In the case of the $S\&P$ index, the *MSGARCH*-t and *EGARCH*-t models have the best performance, while the *MRS-GARCH* also performs quite accurately. Recently, *MS-GARCH* models have been popular for financial time series analysis and have been used in many econometric studies. For example, Chen (2012) used the Bayesian standard, the non-linear threshold and *MS-GARCH* to predict the *VaR*, both before and after the financial crisis. Consequently, the models were developed according to Chen *et al.* (2009), Chen *et al.* (2017), and Sampid *et al.* (2018). To study the asymmetric effects on the conditional mean and the conditional volatility of time series, Chen *et al.* (2009) extended the *MS-GARCH* model. They presented a dual *MS-GARCH* model (*DMS-GARCH*). The main advantages of this model are the capture of leptokurtotic, the clustering of volatility, and an asymmetric correlation between time series and conditional volatility. Chen *et al.* (2017) proposed a general and time-varying smooth transition (*ST*) with conditional heteroscedastic models, and used a second-order logistic function of mean and discourse of variance. The advantage of this model is the greater flexibility of the model's parameters. A Bayesian model *MS-GJR-GARCH*$(1,1)$ with a skewed Student's-t distribution, copula functions, and Extreme Value Theory (*EVT*) was presented by Sampid *et al.* (2018). The main feature of this model are to study the fluctuation and volatility and allow the parameters of the *GARCH* model to fluctuate over time according to a Markov process. In addition, they proposed the selection of thresholds in extreme value theory. Especially in quiet periods and in crisis periods, the empirical study shows that this model can capture *VaR* well. More authors have recently worked on stock markets in order to select the portfolio on uncertain returns with the programming methods. Chen *et al.* (2017) and Zhang *et al.* (2015) have addressed the problem of portfolio selection in an uncertain environment where stock returns cannot be well reflected by historical data, but can be evaluated by experts. They assumed security returns to be given by an uncertain variable. Recently, Ardia *et al.* (2018) perform a large-scale empirical study in order to compare the forcasting performances of single-regime and Markov-switching *GARCH* (*MS-GARCH*) models from a risk management perspective. They find that *MS-GARCH* models yield accurate Value-at-Risk, expected shortfall, and left-tail distribution forecast than their single-regime counterparts for daily, weekly, and ten-day equity log-returns.

Based on the previous literature, we then introduce an asymmetric two-regime *MRS-GARCH*$(1,1)$ model by combining a *GJR-GARCH*$(1,1)$ specification with stable innovations using the regime-switching Markov model introduced by Gray (1996). Since the literature is limited to taking into account the asymmetric characteristics and the heavy tails of financial series, the *GJR-GARCH*$(1,1)$ specification of our *MRS-GJR-GARCH*$(1,1)$ solves the asymmetry problem and uses a stable distribution to take into account the heavy tails observed in financial series. This is of crucial importance to account for leveraging in the stocks markets. We first present the descriptive statistic of the *S&P 500* stock market index, then the estimation of the parameters of our model by the Monte Carlo method. Second, in the same way as Ardia *et al.* (2018), we evaluate the performance of our model using a backtesting procedure, statistical techniques to test the significance of *VaR* violations and several common procedures to assess the quality of risk predictions over two periods: two years before the *COVID-19* pandemic and one year during the *COVID-19* pandemic. Backtesting is useful to identify weaknesses in risk forecasting models and to provide ideas for improvement, but it is not informative about the causes of the weaknesses. Model that do not work well in backtesting need to have their assumptions and parameter estimates questioned. However, backtesting can avoid underestimating *VaR* and thus ensure that a bank has sufficient capital. At the same time, reducing the probability of overestimating *VaR* due to backtesting can lead to excessive conservatism.

In particular, this study is to examine whether the *MRS-GJR-GARCH* model with two-regime using stable distribution, improves the forecasting *VaR* model compared to their single regime counterpart. *VaR* model is widely used in forecasting risk on the stock return volatility, however, there have been more studies on the *VaR* model in this stock index. Thus, we would like to extend the limited empirical research on *VaR* estimation to forecasting *S&P 500* return volatility. The remainder of this paper is organized as follows. Section 2 briefly describes the *MRS-GJR-GARCH*(1,1) model with the stable distribution. Section 3 provides the descriptive statistics of data and empirical results. The last section concludes.

2. Model Specification

2.1. Markov-Switching GJR-GARCH (MS-GJR-GARCH) Model

In order to describe two-state Markov Regime-Switching *GJR-GARCH* model, we start with a *GJR-GARCH* model (Glosten *et al.* (1993)) given in (1) below:

$$\begin{cases} y_t = \xi_t \sqrt{h_t} \quad ; \qquad t = 1, 2, \cdots \\ h_t = \gamma + \big(\alpha_1 + \alpha_2 \mathbb{I}\{y_{t-1} < 0\}\big) y_{t-1}^2 + \beta h_{t-1}; \qquad \gamma > 0, \quad \alpha_1, \alpha_2, \beta \geq 0 \end{cases}$$

with the indicator function

$$\mathbb{I}\{y_{t-1} < 0\} = \begin{cases} 1 & if \quad y_{t-1} < 0, \\ 0 & if \quad y_{t-1} \geq 0, \end{cases}$$

where the conditional variance $h_t = h(\theta_h, \Psi_{t-1})$, with $\theta_h = (\gamma, \alpha_1, \alpha_2, \beta)$ being vector of parameters, and Ψ_{t-1} being the entire past history of the data up to time $t-1$, and ξ_t is a stationary sequence of random variables with mean zero and variance one.

If $\alpha_2 > 0$ then a leverage effect exists, that is negative news has a bigger impact on volatility than positive news. If $\alpha_2 \neq 0$, the news impact is asymmetric. The leverage effect is often described as a falling equity price which leads to an increase in a firm's debt to equity ratio which increases the volatility of returns to equity holders.

Let S_t be an ergodic Markov chain on a finite set $S = \{1, 2\}$ with transition probabilities matrix

$$P = \begin{bmatrix} P_{11} & P_{21} \\ P_{12} & P_{22} \end{bmatrix} = \begin{bmatrix} p & (1-q) \\ (1-p) & q \end{bmatrix},$$

where $P_{ij} = Pr(S_t = i | S_{t-1} = j)$. In this study, the state variable (S_t) takes value 0 or 1 referring to a two-state.

For Markov switching *GJR-GARCH* model, we have

$$h_{it} = \gamma_i + \big(\alpha_{1,i} + \alpha_{2,i}\mathbb{I}\{y_{t-1} < 0\}\big)y_{t-1}^2 + \beta_i h_{t-1}, \quad i = S_t \in \{1, 2\},$$

where h_{it} is a Ψ_{t-1}-measurable function, which defines the filter for the conditional variance and ensures that it is positive, and the indicator function $\mathbb{I}\{\cdot\}$ is defined to be 1 if the condition holds and 0 otherwise. We have $\theta_i = (\gamma_i; \alpha_{1,i}; \alpha_{2,i}; \beta_i)$ with $S_t = i = 1, 2$. To ensure the positivity of the conditional variance we impose the restrictions $\gamma_i > 0$, $\alpha_{1,i} \geq 0$, $\alpha_{2,i} \geq 0$, and $\beta_i \geq 0$.

The degrees of asymmetry in the conditional volatility is governed by the parameter $\alpha_{2,i}$.

2.2. Distribution of models

2.2.1. Normal distribution

The probability density function (*PDF*) of the standard normal distribution can be expressed as

$$f(\eta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\eta^2}, \qquad \eta \in \mathbb{R},$$

which may be maximized with respect to $(\beta; \gamma; \sigma; \delta)$.

2.2.2. Student's t and General Error Distribution

Student t distribution has become a standard benchmark in developing models for asset return distribution because it is able to describe fat tails observed in many empirical distributions. Also, its mathematical properties are well known. The density function of the standardized student-t distribution can be expressed as

Student"s t:

$$f(\eta, v) = \frac{\Gamma(\dfrac{v+1}{2})}{\Gamma(\dfrac{v}{2})\sqrt{\pi(v-2)}} \left(1 + \frac{\eta^2}{(v-2)}\right)^{-\dfrac{v+1}{2}}, \qquad \eta \in \mathbb{R},$$

where $\Gamma()$ is the Gamma distribution. To guarantee the second order moment exists, the constraint of v has to be higher than two. The kurtosis of the distribution is higher for lower v.

The density function of the standardized generalized error distribution (*GED*) can be expressed as

$$f(\eta, v) = \frac{v e^{-\frac{1}{2}|\eta/\lambda|^v}}{\lambda 2^{(1+\frac{1}{v})}\Gamma(\frac{1}{v})}, \qquad \lambda = \left(\frac{\Gamma(1/v)}{4^{1/v}\Gamma(3/v)}\right), \qquad \eta \in \mathbb{R}, \tag{1}$$

where v is the shape parameter which has to be greater than zero.

2.2.3. The α-Stable distribution

Stable random variables were first introduced by Gnedenko and Kolmogrov (See Gnedenko and Kolmogorov (1954)) in a study of the sum of random variables. However, the first formal definition of a stable random variable was given by Feller (1971). Stable distributions were also studied by Zolotarev (1986) and applied to finance by Rachev and Mittnik (2000). The family of stable distributions or α-stable distribution replaces the generally used fat-tailed distribution. Calzolari *et al.* (2014) proposed it. The α-stable distribution has the particularity of overcoming the problem of stability. Moreover, by taking into account asymmetry and heavy tails, the α-stable distribution thus constitutes a generalization of the Gaussian distribution. It is generally said that a random variable x is stably distributed if and only if, for any positive number c_1 and c_2, there is a positive number k and a real number d such that

$$kx + d = c_1 x_1 + c_2 x_2,$$

where x_1 and x_2 are independent variables and have the same distribution as x. Equation (1) describes the property of stability under addition (Tankov (2003)). In particular, if $d = 0$, x is said to be strictly stable. The theoretical foundations of the alpha-stable distribution, according to Calzolari *et al.* (2014), are based on the generalized central limit theorem, in which the finite variance condition is replaced by a much less restrictive condition concerning regular tail behaviour. The α-stable distribution can be described by its characteristic function because this distribution has no close density function, which is the following form.

$$\phi(t) = \begin{cases} \exp\left(i\delta t - \sigma^\alpha |t|^\alpha (1 - i\beta sign(t) tan(\pi\alpha/2)\right), & \alpha \neq 1 \\ \exp\left(i\delta t - \sigma^\alpha |t|^\alpha (1 - i\beta \frac{2}{\pi} sign(t) \log|t|)\right), & \alpha = 1 \end{cases}$$

where

$$sign(t) = \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0, \end{cases}$$

$\alpha \in [0, 2]$ is the index of stability or characteristic exponent that describes the tail-thickness of the distribution (small values indicating thick tails). The scale parameter $\sigma > 0$ measures the spread of the distribution. The location parameter $\delta \in \mathbb{R}$ is a rough measure of the midpoint of the distribution. The skewness parameter β lies in the closed interval $[-1, 1]$ and is a measure of the asymmetry of the distribution.

Calzolari *et al.* (2014) only consider the symmetric α-stable distribution ($\beta = 0$), which is then characterized by (α, σ, δ) and is denoted as $S_\alpha(0, \sigma, \delta)$. Therefore, the standardized symmetric version is $S_\alpha(0, 1, 0)$ with the following characteristic function

$$\phi(t) = \exp\{-|t|^\alpha\}.$$

If $\alpha = 1$ and $\beta = 0$, the stable distribution is the Cauchy distribution. If $\alpha = 2$ and $\beta = 0$, the stable distribution is the normal distribution. If $1 < \alpha < 2$, the most plausible case for financial series, the tails of the stable distribution are larger than those of the normal distribution and the variance is infinite. Stable distributions, as a class, have the attractive feature that the distribution of the sums of random variables of a stable distribution retains the same shape and skewness, although the resulting distribution changes its scale and location parameters. Moreover, they are the only class of statistical distributions with this characteristic. Assuming that the returns follow a stable distribution, the procedure for calculating *VaR* remains unchanged. The quantile must be derived from the standardized stable distribution $S_\alpha(\beta, 1, 0)$.

2.3. Bayesian Inference

The combination of the likelihood function $L(\Psi|I_T)$ and a prior $f(\Psi)$ gives the kernel of posterior density $f(\Psi|I_T)$. We follow in our work, the study of Ardia (2008), for the anterior density $f(\Psi)$, in which their anterior is constructed from independent diffuse antecedents as follows:

$$f(\Psi) = f(\theta_1, \xi_1) \dots f(\theta_K, \xi_K) f(P)$$

$$f(\theta_K, \xi_K) \propto f(\theta_K) f(\xi_K) I\{(\theta_K, \xi_K) \in CSC_k\}(S_t = 1, ..., K)$$

$$f(\theta_K) \propto f_N(\theta_K, \mu_{\theta_k}, diag(\sigma^2_{\theta_k})) I\{(\theta_k) \in PC_k\}(S_t = 1, ..., K)$$

$$f(\xi_K) \propto f_N(\xi_K, \mu_{\theta_k}, diag(\sigma^2_{\xi_k})) I\{\xi_{k,1}, \xi_{k,2} > 2\}(S_t = 1, ..., K)$$

$$f(P) \propto \prod_{i=1}^{K} \left(\prod_{j=1}^{K} p_{i,j} \right) I\{0 < p_{i,j} < 1\},$$

where the vector of model parameters is $\Psi = (\theta_1; \xi_1; \ldots; \theta_K; \xi_K; P)$. In a state S_t, CSC_{S_t} is the covariance-stationarity condition, PC_{S_t} defines the positive condition, the asymmetry parameter is $\xi_{S_t;1}$, $\xi_{S_t;2}$ designates the tail parameter of the skew-Student t-distribution in the state S_t, $f_N(.; \mu; \sum)$ defines the multivariate normal density with mean μ and variance $\sum$.

The likelihood function $L(\Psi|I_T)$ is $L(y_t|\Psi, I_{t-1}) = \prod_{t=1}^{T} f(y_t|\Psi, I_{t-1})$, where $f(y_t|\Psi, I_{t-1})$ is the density of y_t given by its past observations (I_{t-1}), and the model parameters. The conditional density of y_t for the *MRS-GJR-GARCH* model is expressed as follows

$$f(y_t|\Psi, I_{t-1}) = \sum_{i=1}^{K}\sum_{j=1}^{K} p_{i,j} z_{i,t-1} f_D(y_t|s_t = j, \Psi_{t-1}),$$

with

$$f_D(y_t|s_t = k, \Psi_{t-1}) = \frac{1}{\sqrt{2\pi\sigma_t}} \exp\{-\frac{y_t^2}{2\sigma_t}\},$$

where $z_{i;t-1} = P[s_{t-1} = i|\Psi; I_{t-1}]$ is the filtered probability of state i at time $t-1$: the conditional density of in state y_t given by Ψ and I_{t-1} is $f_D(y_t|s_t = k; \Psi; I_{t-1})$. After we obtain the posterior density function, we employ Markov Chain Monte (*MCMC*) for numerical integration. The marginal posterior density function and the state variables are obtained by integrating the posterior density function. We follow Vihola (2012) that samples are produced from the posterior distribution with adaptive *MCMC* algorithm. The benefit is that converge of Markov chain is faster as when it is coercing the acceptance rate, it also learns the shape of the target distribution. This algorithm also guarantees a positive variance and covariance-stationarity of the conditional variance.

2.4. Forecast of Value-at-Risk (VaR)

The *VaR* methodology was introduced in the early 1990s by the investment bank J.P. Morgan to measure the minimum portfolio loss that an institution might face if an unlikely adverse event occurred at a certain time horizon. The *VaR* measures the threshold value such that the probability of observing a loss more massive or equal to it in a given time horizon is equal to α. The *VaR* estimation in $T+1$ at risk level α can be expressed as

$$\mathit{VaR}_{T+1}^{\alpha} = inf\{y_{T+1} \in \mathbb{R} \big| F(y_{T+1}|I_T) = \alpha\},$$

where $F(y_{T+1}|I_T)$ is the one-step ahead cumulative density function (*CDF*) evaluated in y.

The one-step ahead conditional density of y_{T+1} is a mixture of K regime-dependent distribution can be expressed as

$$f(y_t|\Psi, I_{t-1}) = \sum_{k=1}^{K} \pi_{k,T+1} f_D(y_{T+1}|s_{T+1} = k, \Psi, I_T).$$

The mixing weights $\pi_{k,T+1} = \sum_{i=1}^{K} p_{i,k}\eta_{i,T}$, where $\eta_{i,T} = P[s_T = i|\Psi, I_T]$ are the filtered probabilities at time T. For Bayesian estimation given a posterior sample $\{\Psi^m, m = 1, \cdots, M\}$, the predictive *PDF* can be expressed as

$$f\left(y_{T+1}\middle| I_T = \int_{\Psi} f(y_{T+1}|\Psi, I_T) f(\Psi|I_T) d\Psi\right) \approx \frac{1}{M}\sum_{m=1}^{M} f\left(y_{T+1}\middle|\Psi^{[m]}, I_T\right).$$

Assessing the accuracy of *VaR* forecasts from different models is a considerable task because of the importance of *VaR* in risk management. Here we use some tests to examine the accuracy of *VaR* forecasts.

2.4.1. Unconditional coverage test

A well-specified *VaR* model should produce *VaR* forecasts that cover the pre-specified probability. This means that 5% of the time the losses should exceed the *VaR*(0.05). If the number of exceedances substantially differs from what is expected, then the model's accuracy is questionable. If the actual loss exceeds the *VaR* forecasts, this is termed an "exception", which can be presented by the indicator variable q_t as

$$q_t = \begin{cases} 1 & if \quad y_t < VaR_t(\rho), \\ 0 & if \quad y_t \geq VaR_t(\rho). \end{cases}$$

Obviously, q_t is a Bernoulli random variable with probability φ. The Kupiec test (Kupiec (1995)), also known as the unconditional coverage (*UC*) test, is designed to test the number of exceptions based on the likelihood ratio (*LR*) test. The null hypothesis of the *UC* test is $H_0 : \rho = \varphi$. Then the *LR* test of the unconditional coverage (LR_{UC}) is defined as

$$LR_{UC} = -2log\left(\frac{LR^0_{UC}}{LR^1_{UC}}\right) = -2log\left(\frac{\rho^n(1-\rho)^{T-n}}{\hat{\varphi}^n(1-\hat{\varphi})^{T-n}}\right),$$

where L^0_{UC} and L^1_{UC} are the likelihood functions respectively under H_0 and H_1, T is the number of the forecasting samples, n is the number of the exceptions and $\hat{\varphi} = \dfrac{n}{T}$ is the *ML* estimate of the φ under H_1. Under H_0, the LR_{UC} is asymptotically distributed as a χ^2 random variable with one degree of freedom.

2.4.2. Conditional coverage Independence test

Forecasts should be responsive to changes in volatilities, if they are low in some periods and high in others. This means that *VaR* should be low in periods of low volatility and high in periods of high volatility. Thus, exceptions are spread independently over the entire sample period and do not appear in clusters (Sarma *et al.* (2003)). A model that cannot capture the clustering of volatilities will exhibit the symptom of a clustering of exceptions. Kupiec's test cannot verify the clustering of exceptions. To test exception clustering, Christoffersen (1998) designed an independent conditional coverage test (*CCI*) based on *LR*. The null hypothesis of the *CCI* test assumes that the probability of an exception on a given day t is not influenced by what happened the day before. Formally, $H_0 : \varphi_{10} = \varphi_{00}$, where φ_{ij} denotes that the probability of an i event on day $t-1$ must be followed by a j event on day t; $\varphi_{ij} = p(q_t = j \big| q_{t-1} = i)$, where $i; j = 0, 1$. The *LR* statistic of the *CCI* test (LR_{CCI}) can be obtained as

$$LR_{CCI} = -2log\left(\frac{LR^0_{CCI}}{LR^1_{CCI}}\right) = -2log\left(\frac{\hat{\varphi}^n(1-\hat{\varphi})^{T-n}}{\hat{\varphi}_{01}^{n_{01}}(1-\hat{\varphi_{01}})^{n_{00}}\hat{\varphi}_{11}^{n_{11}}(1-\hat{\varphi_{11}})^{n_{10}}}\right),$$

where n_{ij} is the number of observations with value i followed by value $j(i; j = 0; 1)$, $\hat{\varphi}_{01} = \dfrac{n_{01}}{n_{00}+n_{01}}$, and $\hat{\varphi}_{11} = \dfrac{n_{11}}{n_0+n_{11}}$. Under H_0, the LR_{UC} is asymptotically distributed as a χ^2 random variable with one degree of freedom.

2.4.3. Conditional coverage test

The *CCI* test is not complete on its own. Hence, Christoffersen (1998) proposed a joint test: the conditional coverage (*CC*) test, which combines the properties of both the *UC* and *CCI* tests. The null hypothesis of the *CC* test checks both the exception cluster and consistency of the exceptions with *VaR* confidence level. The null hypothesis of the test is $H_0 : \varphi_{01} = \varphi_{11} = \rho$. The *LR* test statistic is obtained as

$$LR_{CC} = -2log\left(\frac{LR^0_{CC}}{LR^1_{CC}}\right) = -2log\left(\frac{\rho^n(1-\rho)^{T-n}}{\hat{\varphi}_{01}^{n_{01}}(1-\hat{\varphi_{01}})^{n_{00}}\hat{\varphi}_{11}^{n_{11}}(1-\hat{\varphi_{11}})^{n_{10}}}\right).$$

Under H_0, LR_{CC} is asymptotically distributed as a χ^2 random variable with two degrees of freedom. It is a summation of two separate statistics, LR_{UC} and LR_{CCI}, as given below:

Table 1. Descriptive statistics of the *S&P 500* log-return

Index	Observation	Min	Mean	Max	Skewness	Kurtosis	St.Dev	Jarque-Bera
S&P 500	2251	-5.544	0.020	3.895	-1.0177	25.3216	0.4591	18357.56

$$
\begin{aligned}
LR_{UC} &= -2\Big[log\big(L^0_{CC}\big) - log\big(L^1_{CC}\big)\Big] \\
&= -2\big[log\big(L^0_{UC}\big) - log\big(L^1_{CCI}\big)\Big] \\
&= -2\Big[log\big(L^0_{UC}\big) - log\big(L^1_{UC}\big) + log\big(L^0_{CCI}\big) - log\big(L^1_{CCI}\big)\Big] \\
&= -2\Big[log\big(L^0_{UC}\big) - log\big(L^1_{UC}\big)\Big] - 2\Big[log\big(L^0_{CCI}\big) - log\big(L^1_{CCI}\big)\Big] \\
&= LR_{UC} + LR_{CCI}.
\end{aligned}
$$

3. Discussion: empirical results and backtesting analysis

3.1. Data and Descriptive statistics

In this sub-section, for the analysis of the value at risk of the stock market index, we choose a sample of the daily performance of *S&P 500* from January 19, 2012, to December 29, 2020, composed of 2251 observations. To evaluate our model before the *COVID-19* period, the *S&P 500* return from January 19, 2012, to December 30, 2019, i.e. 2000 observations, is divided into two groups. The first 1490 observations (January 19, 2012, to December 18, 2017) are used as a sample for estimation, while the remaining 510 observations (December 19, 2017, to December 30, 2019) are considered out-of-sample for the evaluation of the pre-pandemic *COVID-19* forecast. During the evaluation of the *COVID-19* period model, 2251 observations are used. 1999 observations (January 19, 2012, to December 27, 2019) are used as a sample for estimation, while the remaining 252 observations (December 30, 2019, to December 29, 2020) are taken as out-of-sample for the evaluation of forecasts.

If p_t is the price of an asset at time t, then $r_t := 100 \times \log(p_t/p_{t-1})$ is the (log) return. We will show below that some of these returns have heavier tails than a normal model, and we use a stable distribution to describe the returns.
In Table 1, the skewness of *S&P 500* return is -1.0177 suggesting that the distribution of the data is strongly skewed to the left or negatively skewed. The kurtosis of *S&P 500*, we have 25.3216 implying that the distribution of the data is leptokurtic, which means the existence of fat tails. Jarque-Bera test-statistics in Table 1, indicates that the *S&P 500* return is not normally distributed. In Figure 1, we plot

the prices and returns calculated using data from the *S&P 500* stock market index from January 19, 2012 to December 29, 2020.

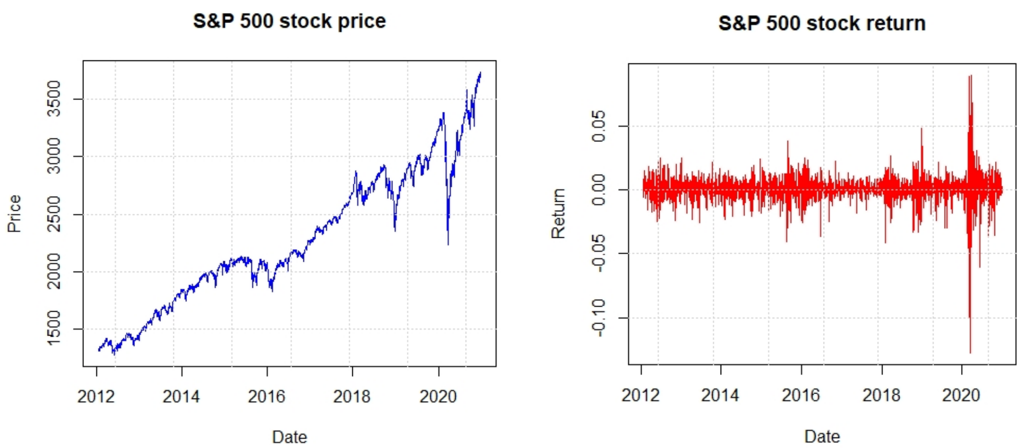

Fig. 1. Daily price (on the left) and daily log return (on the right) of *S&P 500* from 19 January 2012 to 29 December 2020.

3.2. Model parameter estimates

To estimate Markov Switching's *GJR-GARCH* models, we use Thomas Chuffart's *MSG*tool, which is a *MATLAB* toolbox. This toolbox provides a set of functions to simulate and estimate a wide variety of Markov Switching (*MSG*) *GARCH* models. We use it according to the Bayesian method. Parameter estimation of the stable *MRS-GJR-GARCH* model is performed on 2251 observations.

The *S&P 500* return volatility is separated into two regimes, high volatility and low volatility. The high volatility regime is related to high *S&P 500* return deviations, and the low volatility regime is related to small *S&P 500* return volatility. Estimated parameters from Table 2 show that the two regimes have different unconditional volatility levels and volatility persistence. The unconditional volatility of the second regime $(\alpha_{1,2} = 0.0544)$ is higher than that of the first regime $(\alpha_{1,1} = 0.0005)$. Reactions to negative historical returns from the two regimes are different. The first regime is $(\alpha_{2,1} = 0.3314)$, and the second one is $(\alpha_{2,2} = 0.0090)$. Lastly, the volatility persistence of the first regime is 0.9498, and the second regime of 0.9699. The posterior mean stable probabilities indicate that the probability of being in the first regime is 93.62%, whereas being the second one is 6.38%.

Table 2. Parameters estimates of *MRS-GJR-GARCH* stable model.

Regime	Parameters	mean	Standard deviation
Regime 1	$\alpha_{0,1}$	0.0000	0.0000
	$\alpha_{1,1}$	0.0005	0.0017
	$\alpha_{2,1}$	0.3314	0.0441
	β_1	0.7836	0.0194
Regime 2	$\alpha_{0,2}$	0.0000	0.0000
	$\alpha_{1,2}$	0.0544	0.0169
	$\alpha_{2,2}$	0.0090	0.0159
	β_2	0.9111	0.0255
	p_{11}	0.9580	0.0193
	P_{22}	0.6164	0.1838
	Transition matrix $p_{ij} = Pr[s_t = j \vert s_{t-1} = i]$		
	1		2
	1	0.9580	0.0420
	2	0.6164	0.3836
	Stable probabilities		
	State 1		State 2
	0.9362		0.0638

3.3. VaR Backtesting

Backtesting measures the accuracy of *VaR* calculations. The loss forecast is calculated using *VaR* methods and then compared to the actual losses at the end of the next day. The degree of difference between predicted and actual losses indicates whether the *VaR* models underestimates or overestimates the risk. Thus, backtesting examines the data retrospectively and enables the *VaR* model to be evaluated. A common first step in *VaR* backtesting analysis is to plot the return and *VaR* estimates together. Figure 2 shows the results of the one-day *VaR* forecast at the 5% risk level. The solid red line shows the stable *MRS-GJR-GARCH* at two regimes, the blue line represents the single regime *GJR-GARCH* while the realized returns are shown in the black line. The *VaR* backtest consists of testing the performance of the models on 1490 out-of-sample observations, which is used to forecast the two-year *VaR* before *COVID-19*.

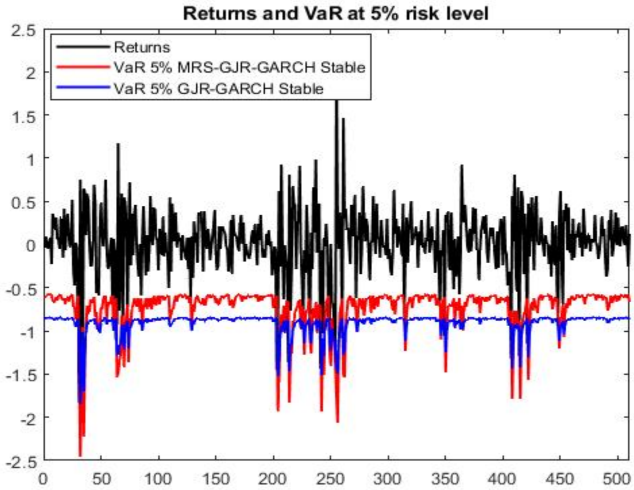

Fig. 2. *S&P 500* returns and *VaR* at 5% risk level from the pre-pandemic *COVID-19*.

A backtesting procedure to evaluate the accuracy of *VaR* forecasts is recommended by the Basel Committee on Banking Supervision (1996). This is generally based on the number of observed violations, i.e. when, during a sampling period, actual losses exceed *VaR*.

We also provide the number of violations and the violation ratios if the forecasting *VaR* value on a given day is higher than the actual data, the violation is occurs. The ratio of the number of observed violations to the expected number of violations is called the violation rate. In this study, violations are expected in 5% of cases. The results have shown in Table 3. To formally evaluate the performance of *VaR* forecast accuracy, we employ three standard *VaR* backtest procedures. The backtesting methods are the proportion of failure (*POF*), conditional coverage (*CC*), and conditional coverage independence (*CCI*) tests. These backtesting procedures basically perform frequency, joint, and independence tests of the *VaR* models. These are based on the comparison between the number of times that losses exceed the *VaR* and the expected number using statistical tests, such as the proportion of failure (*POF*) test of Kupiec (1995), the conditional coverage (*CC*) test and conditional coverage independence (*CCI*) of Christoffersen (1998). The *POF* test is employed to investigate the frequency of failures in *VaR* models. Backtesting a *VaR* model measures the probability of observing a violation or failure in a sample period. However, the effectiveness of any *VaR* model is invalidated if the failures occur in clusters on consecutive days. As a remedy, the conditional coverage (*CC*) test and conditional coverage independence (*CCI*) of Christoffersen (1998) evaluate the correctness of out-of-sample interval forecasts. These tests accept

Table 3. Backtesting *S&P 500* returns with T = 510 observation.

Models	Distributions	NV	Expected	Violation Ratio	*VaR* Volatility
MRS-GJR-GARCH	norm	36	25.5	1.4118	0.1829
	student	14	25.5	0.5490	0.3105
	ged	36	25.5	1.4118	0.1859
	stable	**20**	25.5	**0.7843**	**0.2525**
GJR-GARCH	norm	**27**	25.5	**1.0588**	**0.0866**
	student	11	25.5	0.4314	0.1470
	ged	**26**	25.5	**1.0196**	**0.0880**
	stable	14	25.5	0.5490	0.1195
GARCH	norm	**27**	25.5	**1.0588**	**0.0411**
	student	8	25.5	0.3137	0.0697
	ged	**27**	25.5	**1.0588**	**0.0417**
	stable	14	25.5	0.5490	0.0567

or reject the *VaR* models in terms of their likelihood ratio statistic. According to Christoffersen (1998), *VaR* forecasts are valid if and only if the failure processes fulfills the unconditional coverage and independence test.

The unconditional coverage test is employed to test whether the *VaR* violation rate is statistically different from the confidence level α. The *CC* test improves the unconditional coverage test by not only testing the joint assumption of the unconditional coverage, but also testing whether the probability of *VaR* violation is independent over time.

In this study, *VaR* was estimated over the test window with four different distributions and at one *VaR* confidence level. A violation about 5% of the time is expected for a *VaR* estimate at 95% confidence, and *VaR* failures do not cluster. The lack of independence over time is created by the clustering of *VaR* failures because *VaR* models react slowly to changing market conditions.

Examining Table 3, the *MRS-GJR-GARCH* stable model best estimates the performance of *S&P 500* with the numbers of violations $NV = 20 \in]16;36[$, violation rate $VR = 0.7843 \in [0.7;1.3]$ followed by the *GJR-GARCH* normal, GARCH normal, and *GED* with $NV = 27, VR = 1.0588$. The volatility of the *VaR* of the latter is low than the other. The other models have either $NV \geq 36$ and $VR > 1.3$, but with low *VaR* volatility, or $NV < 16$ and $VR < 0.5$, but with high *VaR* volatility. To confirm our observation, the *POF*, *CC* and *CCI* tests were performed in Table 4.

Although the *GJR-GARCH* normal, *GARCH* normal and *GED* models and the *MRS-GJR-GARCH*(1,1) stable model have acceptable violation ratios, only the *GJR-GARCH* normal and *MRS-GJR-GARCH*(1,1) stable models pass the *POF*, *CC*, and *CCI* tests. Table 4 shows that the *MRS-GJR-GARCH*(1,1) stable model has a 95%

Table 4. *POF*-test, *CC*-test, *CCI*-test and P-value results.

Models		***POF***		***CC***		***CCI***	
		LR_{POF}	**p-value**	LR_{CC}	**p-value**	LR_{CCI}	**p-value**
MRS-GJR-GARCH	normal	4.0577	0.0439	**4.8967**	**0.0864**	**0.8390**	**0.3597**
	student	6.4814	0.0109	7.2174	0.0271	**0.7359**	**0.3909**
	ged	4.0577	0.0439	**4.8967**	**0.0864**	**0.8390**	**0.3596**
	stable	**1.3444**	**0.2463**	**1.4029**	**0.4958**	**0.0586**	**0.8087**
GJR-GARCH	normal	**0.0912**	**0.7627**	**3.7206**	**0.1556**	**3.6294**	**0.0568**
	student	10.932	$9.449e^{-4}$	11.418	$3.315e^{-3}$	**0.4859**	**0.4857**
	ged	**0.0103**	**0.9193**	**4.1032**	**0.1285**	4.093	0.0431
	stable	6.4814	0.0109	7.2174	0.0271	**0.7359**	**0.3909**
GARCH	normal	**0.0912**	**0.7626**	9.9306	$6.975e^{-3}$	9.8394	$1.708e^{-3}$
	student	17.077	$3.589e^{-5}$	17.332	$1.723e^{-4}$	**0.2555**	**0.6132**
	ged	**0.0912**	**0.7626**	9.9306	$6.975e^{-3}$	9.8394	$1.708e^{-3}$
	stable	6.4814	0.0109	7.2174	0.0271	**0.7359**	**0.3909**

confidence level of $LR_{CC} = 1.4029 < 5.99$, $LR_{POF} = 1.3444 < 3.84$, and $LR_{CCI} = 0.0586 < 3.84$, and the *GJR-GARCH* normal model with $LR_{CC} = 3.7206 < 5.99$, $LR_{POF} = 0.0912 < 3.84$, and $LR_{CCI} = 3.6294 < 3.84$. The other models are rejected in the other tests. Values in bold are interpreted as model acceptance, while normal values are interpreted as model rejection.
Given that the health crisis related to the *COVID-19* pandemic has caused shocks in global financial markets, particularly in the *U.S. S&P 500* financial market, financial risk forecasting models are struggling to account for these observed changes in volatility regimes. We, therefore, evaluate our Value-at-Risk model over the one-year period of the *COVID-19* pandemic to show its effectiveness in predicting risks on the *S&P 500* Index. We repeated the simulation in Table 5 and compared our model with a stable distribution to other models. In Table 5, the *MRS-GJR-GARCH*(1,1) model using a stable distribution best estimates the performance of the *S&P 500* with the numbers of violations $NV = 18 \in]6;20[$, Violation Ratio $VR = 1.4286 \in [0.5;1.5]$ followed by *MRS-GJR-GARCH*(1,1) *GED*, with $NV = 6, RV = 0.4762$ and the models GJR-GARCH(1,1) stable, *GARCH*(1,1) stable have $NV = 20$, and $VR = 1.5873$. The last two models have high *VaR* volatility. The other models have either $NV > 20$, and $VR > 1.5$ but with low *VaR* volatility or $NV < 6$, and $VR < 0.5$, but with high *VaR* volatility.

For 95% confidence level, the *MRS-GJR-GARCH* stable has an $LR_{CC} = 4.9469 < 5.99$, $LR_{POF} = 2.163 < 3.84$, and $LR_{CCI} = 2.7839 < 3.84$ in Table 6. Thus, only the *MRS-GJR-GARCH*(1,1) model with stable distribution is accepted in all tests. Although GJR-GARCH(1,1) stable passed the *CC*, *CCI* tests and that the *GARCH*(1,1) with the four distribution has passed only the *CCI* test, but they are all rejected in other tests. Values in bold are interpreted as acceptance of the model whereas normal values are interpreted as a rejection of the model. Other models are also rejected.

Table 5. Backtesting *S&P 500* returns during *COVID-19* period with T = 252 observation.

Models	Distributions	NV	Expected	Violation Ratio	*VaR* Volatility
MRS-GJR-GARCH	norm	27	12.6	2.1429	0.6065
	student	3	12.6	0.2381	1.5470
	ged	6	12.6	0.4762	1.4617
	stable	**18**	12.6	**1.4286**	**0.8373**
GJR-GARCH	norm	29	12.6	2.3016	0.3554
	student	3	12.6	0.2381	0.9064
	ged	4	12.6	0.3175	0.8565
	stable	20	12.6	1.5873	0.4636
GARCH	norm	30	12.6	2.3810	0.2151
	student	5	12.6	0.3968	0.5486
	ged	5	12.6	0.3968	0.5183
	stable	20	12.6	1.5873	0.2806

Table 6. *POF*-test, *CC*-test, *CCI*-test and P-value results.

Models		***POF***		***CC***		***CCI***	
		LR_{POF}	**p-value**	LR_{CC}	**p-value**	LR_{CCI}	**p-value**
MRS-GJR-GARCH	norm	13.24	$2.741e^{-4}$	13.627	$1.099e^{-3}$	**0.3876**	**0.5336**
	student	10.969	$9.264e^{-4}$	11.042	$4.002e^{-3}$	**0.0726**	**0.7876**
	ged	4.4771	0.0344	**4.771**	**0.0920**	**0.2939**	**0.5877**
	stable	**2.163**	**0.1414**	**4.9469**	**0.0843**	**2.7839**	**0.0952**
GJR-GARCH	norm	16.699	$4.381e^{-5}$	16.747	$2.309e^{-4}$	**0.0482**	**0.8262**
	student	10.969	$9.263e^{-4}$	11.042	$4.002e^{-3}$	**0.0726**	**0.7876**
	ged	8.3261	$3.908e^{-3}$	8.4556	0.0146	**0.1296**	**0.7189**
	stable	3.9126	0.0479	**4.2073**	**0.1220**	**0.2948**	**0.5872**
GARCH	norm	18.546	$1.658e^{-5}$	18.607	$9.112e^{-5}$	**0.0601**	**0.8064**
	student	6.1962	0.0128	6.3994	0.04078	**0.2033**	**0.6521**
	ged	6.1962	0.0128	6.3994	0.0408	**0.2033**	**0.6521**
	stable	3.9126	0.0479	4.0272	0.1335	**0.1147**	**0.7349**

Figure 3 shows the results for the *VaR* forecasting at the 5% risk level during the *COVID-19*. The solid red line displays the model of two regimes, the blue line represents the single regime whereas the returns are presented in the black line.

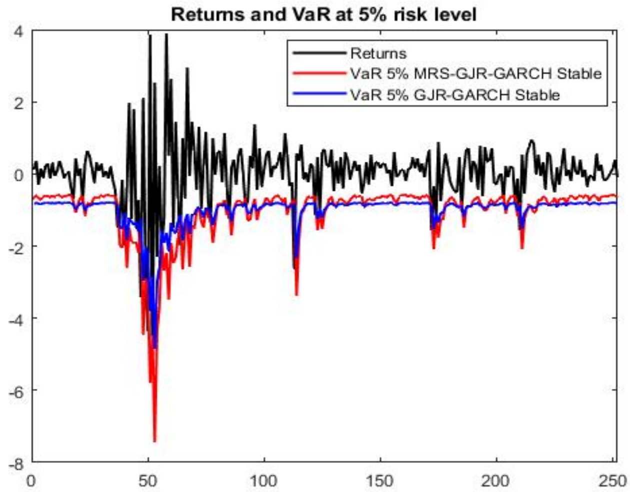

Fig. 3. *S&P 500* returns and *VaR* at 5% risk level during the pandemic *COVID-19*.

3.4. Discussion

We have used a Markov regime switching *GJR-GARCH* with stable distribution to illustrate the relative forecasting performances. We apply the methods of estimation of the Value-at-Risk and of backtesting described in Section 2 on the data set described above. Furthermore, we plot in Figure 1 for *S&P 500* stock his prices, returns computed from data, and *VaR* computed under our models. Our comparisons are based on a backtesting for a single level of confidence one level of confidence 5% of the *VaR*. The performance of each model is given by an average of its results for the *S&P 500* stock. Figure 2 gives the violation ratios for the models on the *S&P 500* stock index for the period before the *COVID-19*. Clearly the other models overestimates or under-estimates the *VaR*. *GJR-GARCH* normal and Markov regime switching *GJR-GARCH* stable models results fluctuate around the target value 5% in pre-pandemic and during the *COVID-19* pandemic, only the Markov regime switching *GJR-GARCH* stable perform well. In terms of average of violation ratios, the *MRS-GJR-GARCH* stable model is the closest to 5% with 0.78%, and 1.42% respectively in pre-pandemic and during the *COVID-19* pandemic. The backtest results over the stock in Table 3 and Table 5 indicate that the *MRS-GJR-GARCH* stable model gives the best results in terms of backtest. The proportion failure, unconditional, and conditional coverage tests give better results for the regime switching model with stable distribution. The results of *CC*, *CCI* and *POF* tests are given in Table 4 and Table 6.

4. Conclusion

The objective of this paper is to study the forecasting performance of the stable model *MRS-GJR-GARCH* in estimating the *VaR* for the *S&P 500* volatility in moments of economic crisis such as the one created by the *COVID-19* pandemic. The results show that *MRS-GJR-GARCH* with a stable distribution proves to be the best fit for the data. The estimated parameter of *MRS-GJR-GARCH* with a stable distribution shows that the two regimes report different levels of unconditional volatility and persistence of the volatility process. Our *MRS-GJR-GARCH* stable model gives an average violation rate on *S&P 500* stocks closer to 5% than the other models, and the model is statistically significant for the *POF*, *CC*, and *CCI* tests for most of the two periods, the normal period i.e. the period before the *COVID-19* pandemic and the period during the *COVID-19* pandemic. A Comparison of the performance of *S&P 500* and *VaR* at a risk level of 5% confirms that the stable two-regime *MRS-GJR-GARCH* model performs better than its single regime counterpart. Thus, this study implies that the two-regime *MRS-GJR-GARCH* model with stable innovation seems to improve the forecasting performance of the *VaR* model on the volatility of the *S&P 500* return.

Further study will be focused on the heavy tails distribution by comparing *GARCH* models and the conditional Extreme Value Theory (*EVT*) specifications to improve the *VaR* forecasting performance to overcome the problem of difficulty of *VaR* models in estimating the heavy-tailed of the return distribution.

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